Advanced Engineering Science
http://publish.mersin.edu.tr/index.php/ades
e-ISSN 2791-8580

Helmholtz equations and their applications in solving physical problems

Davron Aslonqulovich Juraev *[1,2], **Praveen Agarwal** [3], **Ebrahim Eldesoky Elsayed**[3], **Nauryz Targyn** [4]

[1] University of Economy and Pedagogy, Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, Uzbekistan, juraevdavron12@gmail.com
[2] Anand International College of Engineering, Department of Mathematics, Jaipur, India, goyal.praveen2011@gmail.com
[3] Mansoura University, Faculty of Engineering, Department of Electronics and Communications Engineering (ECE), Mansoura, 35516, El-Dakahilia, Egypt, engebrahem16@gmail.com
[4] Kazakh-British Technical University, International School of Economic; Institute of Mathematics and Mathematical Modeling, Department of Mathematical Modeling and Mathematical Physics, Almaty, Kazakhstan, targyn.nauryz@gmail.com

Cite this study: Juraev, D. A., Agarwal, P., Elsayed, E. E., & Targyn, N. (2024). Helmholtz equations and their applications in solving physical problems. Advanced Engineering Science, 4, 54-64

Keywords
The Helmholtz equation
Wave equation
Law of conservation of energy
Physics problems

Research Article
Received: 25.12.2023
Revised: 24.01.2024
Accepted:10.02.2024
Published: 20.02.2024

Abstract

The Helmholtz equation is a well-known concept in the field of physics, particularly when studying problems involving partial differential equations (PDEs) in both space and time. It is a time-independent form of the wave equation and is derived using the method of separation of variables to simplify the analysis. In this research article, we explore the Helmholtz equation and delve into its physical significance. The Helmholtz equation plays a crucial role in solving various physics problems, such as seismology, electromagnetic radiation, and acoustics. It encompasses a wide range of scenarios encountered in electromagnetics and acoustics and is equivalent to the wave equation under the assumption of a single frequency. In the context of water waves, it emerges when the dependence on depth is removed. This often leads to a transition from the study of water waves to more general scattering problems. By employing a cylindrical eigenfunction expansion, we observe that the modes related to the Helmholtz equation decay rapidly as distance approaches infinity. This property enables us to derive asymptotic results in linear water waves based on findings in acoustic or electromagnetic scattering.

1. Introduction

Let's start by discussing the Helmholtz equation. Named after Hermann von Helmholtz, it is a linear partial differential equation that encompasses various important concepts such as the Laplacian, amplitude, and wavenumber. The Helmholtz equation is also known as an eigenvalue equation. To solve the Helmholtz differential equation, one commonly employs the method of separating the variables, especially in polar coordinate systems. This approach helps simplify the complexity of the analysis. The Helmholtz equation frequently arises in the study of physical problems that involve partial differential equations (PDEs) in both spatial and temporal domains. It serves as a time-independent form of the wave equation. In mathematics and physics, the Helmholtz equation plays a significant role, providing a mathematical representation for various phenomena. Its formula is shown in Equation 1:

$$\nabla^2 W + \Lambda^2 W = 0 \tag{1}$$

Here ∇^2 – Lapsasian, Λ – wavenumber, W – amplitude. The unknown function W is defined by $\mathbf{R}^n$ (in practice, the Helmholtz equation is applied for $n = 1, 2, 3$).

The application of the Helmholtz equation extends to various disciplines, including seismology, acoustics, and electromagnetic radiation.

Seismology is the scientific study of earthquakes and the propagation of elastic waves. It encompasses the analysis of phenomena such as tsunamis and volcanic eruptions, taking into account seismic sources. Seismic waves can be categorized into two types: body waves and surface waves. Body waves, including P-waves (primary waves) and S-waves (secondary or shear waves), can travel through different layers of the Earth and are the fastest waves, often recorded first by seismographs. On the other hand, surface waves are confined to the Earth's surface. Understanding these waves is fundamental to comprehend seismology and related concepts.

In the context of solving the Helmholtz equation, various scholars have contributed to solving it for different shapes. For example, Simeon Denis Poisson applied the equation to solve problems involving rectangular membranes, while Gabriel Lame tackled equilateral triangle shapes. Alfred Clebsch utilized the Helmholtz equation to address circular membrane problems.

In thermodynamics, the Helmholtz function is defined as the thermodynamic function of a system, representing the difference between its internal energy and the product of the system's temperature and entropy. Additionally, the Helmholtz free energy corresponds to the work extracted from a system while keeping the temperature and volume constant. In contrast, the Gibbs free energy represents the maximum reversible work extracted from the system while maintaining constant temperature and pressure.

2. Helmholtz's law of conservation of energy

Energy, a highly complex concept in modern physics, is governed by the fundamental principle of the conservation of energy. In collaboration with an expert, we will explore challenges related to this fundamental law and uncover the individuals who made significant discoveries in this field. Physics, as a discipline, seeks to comprehend the fundamental laws governing the material world. While figures such as Archimedes, Newton, and Einstein are widely recognized, numerous scientists have contributed to the advancement of modern science, laying the foundation for human civilization. The progress achieved in understanding the nature of energy and its laws, particularly within the branch of mechanics, lies at the core of this development, as it represents a branch of physics that allows for direct observation and experimentation.

The pervasive nature of the law of conservation of energy often goes unnoticed. In the realm of mechanics, it operates within closed systems influenced solely by conservative forces such as gravity and elasticity. These forces rely on the initial and final position of an object, independent of its trajectory. Consequently, the energy of bodies undergoes transformation, transitioning between kinetic and potential states. This basic formulation of the law of conservation of energy pertains specifically to mechanical systems. It is a fundamental law of nature, empirically established, claiming the existence of a scalar physical quantity called energy in an isolated physical system. This quantity is determined by the system's parameters and remains constant over time. As the law of conservation of energy is a general principle applicable universally, it is often referred to as the principle of conservation of energy, reflecting its broader scope beyond specific quantities and phenomena. Historically, the law of conservation of energy was first observed within the field of mechanics. While Galileo initially applied this concept intuitively, recognizing that the speed attained during the fall of an object allowed it to rise back to its original height, it was Huygens who extended this understanding to the center of gravity within a system of falling bodies. In 1695, Leibniz formulated the principle, stating that the product of force and distance yields an increase in "living force" (vis viva). Although Newton did not assign great importance to this concept, Johann Bernoulli frequently emphasized the conservation of "living forces" (conservatio virium vivarum), highlighting that the disappearance of living force does not entail a loss of work but rather a transformation into another form. By 1800, it was firmly established that, in systems of material points subjected to central forces, living force depended solely on the configuration of the system and a specific function of the forces related to that configuration. In 1807, Thomas Young introduced the term "energy" to describe living force, and in 1826, Jean Victor Poncelet introduced the concept of "work." Additionally, it became clear that the construction of a perpetual motion machine was unattainable. This realization, prevalent by the end of the 18th century, prompted the French Academy to reject further hypothetical solutions to this problem. The true implications of this negative perspective emerged more prominently in the 19^{th} century.

Hermann Helmholtz, whose broad intellect comprehended the universal significance of the law of conservation of energy, played a pivotal role in shaping our understanding of this principle. Initially approaching the concept from the field of medicine, Helmholtz independently arrived at conclusions similar to those of Julius Mayer, whose work he was initially unaware of. In a small publication in 1845, Helmholtz correctly identified an error in the analysis of renowned chemist Justus Liebig, highlighting the impossibility of equating unconditionally the heat of combustion in an animal's body with the heat of combustion of its constituent chemical elements. In this article, Helmholtz briefly outlined the consequences of the law of conservation of energy across various areas of physics. He introduced the concept of "potential energy" to mechanics and provided energy equations for gravitational, static electric, and magnetic fields. Helmholtz also discussed the energy perspective in relation to galvanic and thermal elements' current production and elucidated electromagnetic phenomena, including induction.

The work of Helmholtz has had a profound impact on our understanding and calculations of energy. Nowadays, when calculating the energy of gravitational fields or electric fields, we rely directly on the principles established by Helmholtz. Following Helmholtz's contributions, the concept of energy became central in physics, influencing the evaluation of new theories. In 1890, some scientists, such as Wilhelm Ostwald, went as far as making energy the cornerstone of their worldview, known as "energetics." They sought to derive other physical laws from the concept of energy and even challenged the second law of thermodynamics by denying the distinction between reversible and irreversible processes. For instance, they placed the transfer of heat from higher to lower temperatures on the same level as the fall of bodies in a gravitational field. While there were debates, Plaik's arguments against this viewpoint were met with limited success, whereas Ludwig Boltzmann effectively countered it using atomic theory and statistics. Eventually, misconceptions surrounding the concept of energy disappeared with the passing of its proponents. The concept of energy has also found its way into technology, where each machine is evaluated based on its energy balance, quantifying how much input energy is converted into the desired form. Today, the concept of energy is part of the knowledge base of every educated person. In Newtonian mechanics, kinetic energy holds a special significance as it is inherently tied to motion and combines with other forms of energy. With the advent of the theory of relativity, this special form of energy diminishes. Instead, every form of energy is multiplied by a factor dependent on speed. This shift in perspective is closely related to the principle of inertia of energy. Reducing any form of energy solely to the inertia of bodies or vice versa would create a circular argument. The law of inertia of energy is often expressed by stating that the mass of a body is equal to its energy at rest divided by the square of the speed of light. Thus, the law of conservation of mass has limited relevance. The gain or loss of heat or work, such as through compression, can alter the mass of a body. Chemical reactions, which release heat, cause a minute decrease in total mass among the participants, although this reduction is difficult to measure accurately. The law of inertia of energy fills the gap that previously existed in the definition of energy. By considering an arbitrarily chosen state as the zero point of energy, the conversion of inert mass to energy becomes a well-defined relationship. This correspondence is exemplified by the fact that an electron and positron can be entirely converted into radiative energy [1-3].

The most important aspect of the law of conservation of energy is that energy neither appears nor disappears. Instead, it can be transferred from one body to another and transformed from one form to another. This law serves as a universal principle that governs all phenomena. According to physicists, if the Universe had a certain amount of energy at its creation, that same amount of energy persists to this day. The law of conservation of energy is considered one of the fundamental laws of nature. When considering a closed system of bodies–a system that does not exchange energy with external entities–energy is conserved. This implies that if we calculate the energy in its various forms for all the bodies comprising a closed system at a specific moment, that total energy remains unchanged at any other point in time, as long as the system remains closed. Energy can move between bodies or across different forms, but its total quantity remains constant. If, while monitoring a specific body within a closed system and calculating its energy in various forms, we find that the energy has decreased, this indicates that it has transitioned into another form or has been transferred to another body.

The formula for Helmholtz's law of conservation of energy is shown in Equation 2.

$$W = W_P + W_k = const. \tag{2}$$

Here W_p – potential energy, W_k – kinetic energy.

Scientists have been intrigued by the patterns of interaction among physical bodies since ancient times. However, they struggled to express these interactions in the form of a formula or even a principle. It was not until the middle of the 17th century when René Descartes attempted to address this in his work "The Elements of Philosophy". Descartes observed that when one body collides with another, it can only transfer an equal amount of motion that it loses in the process. This idea was further developed by Leibniz, who introduced the concept of "living force," now known as kinetic energy. Mikhailo Lomonosov supported this line of reasoning in his "universal natural law," but these formulations were more like principles rather than precise laws with formulas.

It wasn't until the mid-19th century that physicists transitioned from the concept of "living force" to "kinetic energy," thanks to their accumulated experience with thermal and electrical machines. James Joule and Robert Mayer made significant experimental contributions to the understanding of this law. The most comprehensive mathematical formulation was provided by Hermann Helmholtz, who introduced the concept of potential energy and generalized the law of conservation of energy to all branches of physics, including those that didn't exist during his time, such as the theory of relativity and quantum mechanics. This fundamental law is widely employed to solve practical problems across various fields. Let's now explore some examples of problem-solving using the law of conservation of energy.

Problem 1. A certain body was thrown up vertically with an initial speed of 20 m/s. How high will it rise? Do not take air resistance into account when solving the problem.

Solution: The kinetic energy obtained during the throw will gradually be converted into potential energy (Equation 3):

$$W_p = W_k \tag{3}$$

That is (Equation 4):

$$mgh = \frac{mV^2}{2} \tag{4}$$

Where:
m –body weight;
V –initial speed;
g –free fall acceleration;
h –lifting height.

After transformations from Equation (4) we obtain the formula for the lifting height and by calculating we obtain Equation 5:

$$h = \frac{V^2}{2g} = \frac{20^2}{2 \cdot 9{,}8} = 20{,}4m. \tag{5}$$

Answer: The body will rise to a height of 20,4 m.

Problem 2. The spring was stretched by 30 cm. It is known that it received a potential energy of 48 J. What is the stiffness of the spring?

Solution: The formula for the potential energy of an elastically deformed body has the following form (Equation 6):

$$W_p = \frac{kx^2}{2} \tag{6}$$

After transformations from Equation (6) we obtain the formula for the lifting height and by calculating we obtain Equation 7:

$$k = \frac{2W_p}{x^2} = \frac{2 \cdot 48}{0{,}3^2} = 106{,}66 \text{ N/m} \tag{7}$$

Answer: the spring stiffness is 106,66 N/m.

3. Fundamental Solutions to the Helmholtz Equation

The Helmholtz equation naturally appears from general conservation laws of physics and can be interpreted as a wave equation for monochromatic waves (wave equation in the frequency domain). The Helmholtz equation can also be derived from the heat conduction equation, Schrodinger equation, telegraph and other wave-type, or evolutionary, equations. The Helmholtz equation is used in the study of stationary oscillating processes. If $k = 0$, then Equation (1) becomes the Laplace equation.

It is known that the Helmholtz equation in different spaces has different fundamental solutions describing a certain physical process. The Equation (1) has the following fundamental solutions (Equations 8-11) for $n = 1, 2, 3$

$$\text{For } n = 1, \; W = \frac{ie^{ik|\xi-\eta|}}{2k} \tag{8}$$

$$\text{For } n = 2, \; W = -\frac{i}{4} H_0^{(1)}\left(k|\xi-\eta|\right) \tag{9}$$

$$\text{For } n = 1, \; W = \frac{e^{ik|\xi-\eta|}}{4\pi|\xi-\eta|} \tag{10}$$

$$\text{Finally, for general } n, \; W = c_d k^p \frac{H_0^{(1)}\left(k|\xi-\eta|\right)}{2k} \tag{11}$$

where $p = \frac{n-2}{2}$ and $c_d = \frac{1}{2i(2\pi)^p}$.

The fundamental solutions for different factorizations of the Helmholtz equation exhibit a remarkable similarity. By leveraging these fundamental solutions, approximate solutions for the Cauchy problem can be constructed, where the solution is presented explicitly. This involves the construction of the Carleman matrix, which enables the discovery of a regularized solution. For further details, you may refer to the cited source [4-39].

4. Problems resulting in the Helmholtz equation

To the Helmholtz Equation 12:

$$\Delta W(\zeta_1,\zeta_2,\zeta_3) + \Lambda^2 W(\zeta_1,\zeta_2,\zeta_3) = 0 \tag{12}$$

Where $W(\zeta_1,\zeta_2,\zeta_3)$ is a scalar function describing the corresponding field in the ζ_1,ζ_2,ζ_3 coordinate system; Λ is the wave number, a wide class of problems associated with steady-state oscillations (mechanical, acoustic, electromagnetic, etc.) and diffusion processes is reduced.

Steady-state oscillations. Let us consider, as an example, a membrane S fixed along the boundary C and oscillating under the action of forces periodic in time. The corresponding Equation 13:

$$\Delta\overline{N} = \frac{1}{\sigma^2}\frac{\partial^2\overline{N}}{\partial t^2} - F_0(\xi,\eta)\cos(\omega\tau) \tag{13}$$

When studying periodic processes, it is convenient to use complex functions, replacing Equation (13) with the Equation 14.

$$\Delta\dot{N} = \frac{1}{\sigma^2}\frac{\partial^2\dot{N}}{\partial t^2} - F_0(\xi,\eta)\exp(j\omega\tau) \tag{14}$$

Function $\dot{N}$ is the real part of function N from (7) steady oscillations can be Equation 15.

$$N = W\exp(j\omega\tau) \tag{15}$$

where $j = \sqrt{-1}$.

For the amplitude of steady oscillations T we obtain Equation 16.

$$\Delta W + \Lambda^2 W = -F_0(\xi,\eta), \ \left(\Lambda = \frac{\omega}{c}\right), \tag{16}$$

to which you need to add a condition Equation 17.

$$W|_C = 0 \tag{17}$$

If the membrane circuit C is not fixed, but performs periodic oscillations with the same frequency ω Equation 18.

$$W|_C = f_0\exp(j\omega\tau) \tag{18}$$

then for the function v on the contour C there is an inhomogeneous boundary condition Equation 19.

$$W|_C = f_0 \tag{19}$$

Indeed, steady-state oscillation problems are not limited to the Helmholtz equation alone, but are also relevant in the fields of acoustics and electromagnetic field theory. Additionally, one frequently encounters such problems inhomogeneous media, especially in piecewise-homogeneous media where distinct regions violate homogeneity. This range of questions also encompasses problems in the theory of diffraction.

Gas diffusion in the presence of decay and chain reactions. One specific area where steady-state oscillations arise is in the study of gas diffusion, particularly in cases involving decay reactions and chain reactions. For instance, during the diffusion of certain gases like radium emanation, decomposition reactions of the diffusing gas molecules take place. The rate of these decomposition reactions is typically assumed to be proportional to the gas concentration, equivalent to the presence of negative gas sources. When formulating the diffusion Equation 20 for a stationary diffusion process in a homogeneous medium, one arrives at a specific equation to describe the behavior.

$$D\Delta W + cW = 0, \quad (c < 0), \tag{20}$$

Where D is the diffusion coefficient. Of greatest interest is the case with $c < 0$, corresponding to diffusion in the presence of chain reactions leading to the multiplication of diffusing particles. In the stationary case we obtain the Equation 21.

$$\Delta W + cW = 0, \quad (c < 0), \tag{21}$$

since the chain reaction is equivalent to the presence of sources of diffusing substance proportional to the concentration $W(\xi,\eta,\zeta)$ [40-42].

5. Methods for solving Helmholtz equations

There exist numerous methods for solving boundary value problems described by Helmholtz differential equations. To attain effective solutions, some of the commonly used methods include: potential theory, eigenfunction expansion, variable separation methods, integral transformations, and the method of conformal mappings. These methods, which are also widely employed in solving Laplace and Poisson problems, are recommended for solving various types of elliptic equations.

One of the most prevalent analytical methods for solving Helmholtz equations is the method of separation of variables. This method will be further discussed below.

Variable separation method. Let us construct solutions to the Helmholtz equation with separated variables in specific coordinate systems.

Rectangular coordinates (ξ,η,ζ). Helmholtz equation in Cartesian coordinates Equation 22.

$$\Delta W + \Lambda^2 W \equiv \frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 W}{\partial \eta^2} + \frac{\partial^2 W}{\partial \zeta^2} + \Lambda^2 W = 0 \tag{22}$$

has a separated variable solution. Let us highlight a special solution in Cartesian coordinates Equation 23.

$$\Omega^{(\pm)}(\xi,\eta,\zeta;\lambda,\beta,\Lambda) = \exp(\pm a\zeta + i\lambda\xi + i\beta\eta) \tag{23}$$

which includes plane wave functions; $a = \sqrt{\lambda^2 + \beta^2 + \Lambda^2}$; $0{,}5\pi > \arg a \geq -0{,}5\pi$; λ, β are arbitrary complex separation constants.

Cylindrical coordinates (ρ,ζ,φ). The Helmholtz equation in cylindrical coordinates has separated variable solutions, which contains special solutions (Equation 24-26):

$$\Omega_m^{(2)}(\rho,\zeta,\varphi;\lambda,\Lambda) = I_m(W\rho)\exp(i\lambda\xi + im\varphi) \tag{24}$$

$$\tilde{\Omega}_m^{(2)}(\rho,\zeta,\varphi;\lambda,\Lambda) = \tilde{K}_m(W\rho)\exp(i\lambda\xi + im\varphi) \tag{25}$$

$$\Psi_m^{(\pm 2)}(\rho,\zeta,\varphi;\lambda,\Lambda) = J_m(\lambda\rho)\exp(\mp i\lambda\xi + im\varphi) \tag{26}$$

Equation (24) and Equation (26) are regular in any finite region of space, Equation (25) have a singularity on the $O\zeta$ axis and satisfy the condition at infinity.

Spherical coordinates (r,θ,φ). The Helmholtz equation in spherical coordinates has a separated variable solution. In the future we will use special spherical solutions (Equation 27-28):

$$\Omega_{mn}^{(3)}(r,\theta,\varphi) = j_n(\Lambda r)Q_n^m(\cos\theta)\exp(im\varphi) \tag{27}$$

$$\Omega_{mn}^{(3)}(r,\theta,\varphi) = h_n^{(1)}(\Lambda r)Q_n^m(\cos\theta)\exp(im\varphi), \\ n = 0,1,\ldots;\ \ m = 0,\pm 1,\ldots,\pm n. \tag{28}$$

Equation (27) is regular in any finite region of space, Equation (28) satisfy the condition at infinity.

Coordinates of the prolate spheroid (ψ,ϕ,φ). The Helmholtz equation in the coordinates of a prolate spheroid has solutions with separated variables. Let's consider solutions with separated variables of a special type (Equation 29-30).

$$\Omega_{mn}^{(4)}(\psi,\phi,\varphi; f) = \Psi_{|m|,n}^{(1)}(\psi,\phi)\Phi_{mn}^{(1)}(f,\phi)\exp(im\varphi) \tag{29}$$

$$\tilde{\Omega}_{mn}^{(4)}(\psi,\phi,\varphi; f) = \Psi_{|m|,n}^{(3)}(\psi,\phi)\Phi_{mn}^{(1)}(f,\phi)\exp(im\varphi), \\ f = c\Lambda;\ \ n = 0,1,\ldots;\ \ m = 0,\pm 1,\ldots,\pm n. \tag{30}$$

Equation (29) is regular in any finite region of space, Equation (30) satisfy the condition at infinity.

Coordinates of an oblate spheroid (ψ,ϕ,φ). For the coordinates of an oblate spheroid, we consider solutions with separated variables of the form in oblate spheroidal coordinates and highlight the harmonic functions (Equation 31-32).

$$\Omega_{mn}^{(5)}(\psi,\phi,\varphi; f) = \Psi_{|m|,n}^{(1)}(-if, i\psi)\Phi_{mn}^{(1)}(-if,\phi)\exp(im\varphi) \tag{31}$$

$$\tilde{\Omega}_{mn}^{(5)}(\psi,\phi,\varphi; f) = \Psi_{|m|,n}^{(3)}(-if, i\psi)\Phi_{mn}^{(1)}(-if,\phi)\exp(im\varphi), \\ n = 0,1,\ldots;\ \ m = 0,\pm 1,\ldots,\pm n. \tag{32}$$

Elastic waves refer to mechanical disturbances that propagate through an elastic medium at a finite speed. These disturbances are caused by sources known as elastic wave sources, which act upon the elastic medium. There are two types of elastic waves based on the motion of particles in the medium. A longitudinal wave occurs when particles oscillate in the same direction as the wave's propagation. On the other hand, a transverse wave occurs when particles oscillate perpendicular to the wave's propagation. In liquids and gases, elastic waves are always longitudinal, whereas in solids, both longitudinal and transverse waves can propagate. When traveling through an elastic medium, mechanical disturbances associated with wave sources transfer energy. Hence, these waves are referred to as traveling waves. The speed at which disturbances propagate in the medium is known as the wave speed or phase speed. This wave speed depends on the density and elastic properties of the medium.

A ray is a line that coincides with the tangent at each point and follows the direction of wave propagation. The collection of points where the oscillation phase of the medium's particles has the same value is referred to as the wave surface. In a homogeneous medium, the wave surfaces are perpendicular to the rays. Different types of waves, such as plane, spherical, and cylindrical waves, are distinguished based on the shape of their wave surfaces.

The equation for a plane wave propagating along the $O\xi$ axis (in the positive direction) is shown in Equation 33:

$$z = g\left(\tau - \frac{\xi}{l}\right) \tag{33}$$

If the wave propagates in the negative direction of the Ox axis, then (Equation 34):

$$z = g\left(\tau + \frac{\xi}{l}\right) \tag{34}$$

If the vibrations of particles in a wave are harmonic, then the wave is called harmonic or monochromatic. The equation of a plane harmonic wave traveling along the Ox axis can be written as (Equation 35):

$$z = B\cos(\sigma\tau - \vec{\Lambda}\vec{\rho} + \chi_0) \tag{35}$$

Here A is the amplitude of oscillations in the wave, $\sigma = 2\pi g = \frac{2\pi}{T}$ is the cyclic frequency of the wave, $\Lambda = \frac{\sigma}{l} = \frac{2\pi}{lT}$ is the wave number, $\Xi = \sigma\tau - \Lambda\xi + \chi_0$ is the phase of the wave.

The distance over which the wave propagates in a time equal to the oscillation period is called the wavelength λ (Equation 36):

$$\lambda = lT = \frac{l}{g} \tag{36}$$

Taking this into account, the wave number can be represented as (Equation 37):

$$\Lambda = \frac{2\pi}{\lambda} \tag{37}$$

In the case when a plane wave propagates in an arbitrary direction, its equation has the form (Equation 38):

$$z = B\cos(\sigma\tau - \vec{\Lambda}\vec{\rho} + \chi_0) \tag{38}$$

Here $\vec{\Lambda}$ is the wave vector. Its modulus is equal to wave number Λ, and its direction coincides with the direction of wave propagation at a point with radius vector ρ.

Exponential form of writing the plane wave Equation 39:

$$\tilde{z} = Be^{i(\sigma\tau - \vec{\Lambda}\vec{\rho} + \chi_0)}, \quad (z = \operatorname{Re}\tilde{z}). \tag{39}$$

Equation of a divergent spherical wave (Equation 40):

$$z = B(\rho)\,\mathrm{g}\left(\tau - \frac{\rho}{l}\right) \tag{40}$$

In the case of a monochromatic spherical wave (Equation 41):

$$z = B(\rho)\cos(\sigma\tau - \Lambda\rho + \chi_0) = B(\rho)e^{i(\sigma\tau - \Lambda\rho + \chi_0)} \tag{41}$$

The differential equation describing the propagation of waves in a homogeneous isotropic non-absorbing medium with speed l is called the wave equation and has the form (Equation 42):

$$\Delta z = \frac{1}{l^2}\frac{\partial^2 z}{\partial \tau^2} \tag{42}$$

where $\Delta z = \frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2} + \frac{\partial^2 z}{\partial \zeta^2}$ is the Laplace operator.

If the wave is harmonic, then $\frac{\partial^2 z}{\partial \tau^2} = -\sigma^2 z$ and the wave equation takes the form (Equation 43):

$$\Delta z + \Lambda^2 z = 0 \tag{43}$$

The equation discussed in this article is known as the Helmholtz equation. The characteristics of the wave, such as its amplitude, initial phase, and frequency, are determined by the oscillations occurring in the wave source. As mentioned earlier, the phase speed of the wave relies on the physical properties of the medium in which it propagates. These aspects are further explored in references [42-45]. Some similar problems were considered in works [44-46].

6. Conclusion

This article focuses on the Helmholtz equation, which is an elliptic-type equation with significant physical implications. Physicists commonly analyze both two-dimensional and three-dimensional cases of the Helmholtz equation. Various approaches for solving this equation are presented. By utilizing the fundamental solutions of the Helmholtz equation, approximate solution so for the Cauchy problem and matrix factorizations in different spaces have been constructed. Currently, renowned foreign scientists continue to study solutions to the Helmholtz equation and have made notable advancements in developing new solution methods.

Acknowledgement

We sincerely thank the editor and reviewers for their diligent efforts in enhancing our manuscript. Their valuable contributions and commitment to excellence have greatly improved the quality and clarity of our work. We are immensely grateful for their insightful comments, constructive feedback, and meticulous attention to detail. Their support and guidance throughout this process have been invaluable.

Funding

This research received no external funding.

Author contributions

Davron Aslonqulovich Juraev: Conceptualization, Methodology, Software. **Praveen Agarwal:** Data curation, Writing-Original draft preparation, Software, Validation. **Ebrahim Eldesoky Elsayed:** Visualization, Investigation, Writing-Reviewing and Editing. **Nauryz Targyn:** Visualization, Investigation, Writing-Reviewing and Editing.

Conflicts of interest

The authors declare no conflicts of interest.

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