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The role and essence of ill-posed problems for solving various applied problems

Davron Aslonqulovich Juraev*[1], Ebrahim E. Elsayed [2], Juan D. Bulnes [3], Praveen Agarwal [4]

[1]University of Economics and Pedagogy, Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, Uzbekistan, juraevdavron12@gmail.com
[2]Mansoura University, Faculty of Engineering, Department of Electronics and Communications Engineering, Mansoura, Egypt, engebrahem16@gmail.com
[3]Federal University of Amapá, Department of Exact Sciences and Technology, Brazil, bulnes@unifap.br
[4]Anand International College of Engineering, Department of Mathematics, Jaipur, India, goyal.praveen2011@gmail.com

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Abstract
In this paper, the history of the appearance of ill-posed problems and the role and essence of these problems in solving various problems of equations of mathematical physics are described in detail. At the present time, the theory and application of ill-posed problems in various fields of science is rapidly developing. When solving ill-posed problems, we will construct a Carleman function and, on its basis, we will find in an explicit form an approximate solution of the problem.

Introduction

Many problems of an applied nature, such as geo- and biophysical, electrodynamic, gas-, hydro- and aerodynamic, problems of plasma physics, etc., are reduced to the equations of mathematical physics. In fact, the very construction of an equation of mathematical physics, which adequately describes certain physical laws of the world around us, is a solution to a certain problem, which it is natural to call "inverse". The researcher observes a phenomenon and tries to construct an equation whose solution has the observed properties. Usually, the resulting equations are based on physical laws that allow us to formulate the general form of differential relations. As a rule, they contain a certain number of arbitrary functions that determine the properties of the physical medium. If the properties of the medium are known, then the equation of mathematical physics, combined with the boundary and initial conditions, makes it possible to predict the development of a physical phenomenon in the space-time region. This is a classic problem for the equations of mathematical physics. In the theory of inverse problems such problems are called "direct". In modern natural science, the following inverse problems very often arise: the general form of the equation of mathematical physics is known, but the characteristic properties of the medium are not known, they must be determined from the observed solutions of the equation. A typical situation is when direct measurements inside a certain area are impossible for one reason or another, however, indirect observation and qualitative and quantitative measurements of physical fields at the boundary or outside this area are possible. In mathematical terms, such problems should ensure the correctness of the problem statement.

The concept of the correctness of a problem statement in mathematical physics was formulated at the beginning of the 20th century by the famous French mathematician J. Hadamard [8]. A problem of mathematical physics is called well-posed if the following conditions are met:

1) the solution of the problem exists;
2) the solution of the problem is unique;
3) the solution of the problem continuously depends on the data of the problem.

Having formulated the concept of correctness, J. Adamar gave an example of an ill-posed problem for an equation of mathematical physics, which, in his opinion, did not correspond to any real physical formulation. J. Hadamard showed this on the example of the Cauchy problem for the Laplace equation, which has become a classic example of an ill-posed problem. The need to consider problems of mathematical physics that are incorrect in the classical sense (according to Hadamard) in connection with the problems of interpreting geophysical observational data was first indicated in 1943 by the twice Hero of Socialist Labor, Academician of the USSR Academy of Sciences A.N. Tikhonov. He showed that if the class of possible solutions is reduced to a compact set, then the existence and uniqueness of the solution implies its stability. Ways of development of the theory and methods for solving ill-posed problems are associated with the names of prominent mathematicians A.N. Tikhonov, M.M. Lavrentiev, V.K. Ivanov, as well as with the scientific mathematical schools they created, which largely determined the development of theories and applications of ill-posed problems. A large number of problems in mathematical physics that do not satisfy the Hadamard correctness conditions are reduced to an operator equation of the first kind.

Since the problems of mathematical physics describe real processes in nature, they must satisfy certain requirements. The stability requirement means that any physically defined process must continuously depend on the initial and boundary conditions and on the inhomogeneous term in the equation, i.e. should be characterized by functions that change little with small changes in the initial data. Such processes are not physically defined. Stability is also important for the approximate solution of problems. Among mathematical problems, a class of problems stands out, the solutions of which are unstable to small changes in the initial data. They are characterized by the fact that arbitrarily small changes in the initial data can lead to arbitrarily large changes in the solutions. Problems of this type are, in essence, ill-posed. They belong to the class of ill-posed problems.

Development of ill-posed problems in modern mathematical sciences

Tasks that do not satisfy all of the above requirements 1)–3) are, according to Hadamard, incorrectly delivered. In 1926, T. Carleman (see, for instance [3], p. 41) constructed a formula that connects the values of the analytic function of a complex variable at the points of the region with its values on a piece of the boundary of this region. The construction of the Carleman function makes it possible in these problems to construct a regularization and obtain an estimate of the conditional stability. It is known that the Helmholtz equation in different spaces has a fundamentally different solution. In the future, using the construction of constructing a fundamental solution, we will construct an approximate solution for the Helmholtz equation. MM. Lavrent'ev, in his works on the Cauchy problem for the Laplace equation and for some other ill-posed problems of mathematical physics, indicated a method for distinguishing the correctness class and developed stable methods for solving them ([3-6]). M.M. Lavrent'ev proposed the construction of a regularized solution of the Cauchy problem for the Laplace equation using the Carleman function.

Moreover, in the 1977s, Sh. Yarmukhamedov pointed out the construction of a family of fundamental solutions parametrized by an entire function with certain properties [7]. This construction is used to construct explicit formulas that restore solutions of elliptic equations in a domain from their Cauchy data on a piece of the domain boundary. Such formulas are also called Carleman formulas. The multidimensional Carleman formula was constructed by L.A. Aizenberg [11].

In unstable problems, the image of the operator is not closed, therefore, the solvability condition cannot be written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on a part of the boundary of a domain, the solution is usually unique, the problem is solvable for an everywhere dense data set, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than the theory of solvability of the Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov and N.N. Tarkhanov [10]. An analogue of the Carleman formula for one class of elliptic systems with constant coefficients on the plane is considered in the work of E.V. Arbuzov and A.L. Bukhgeim [12]. The construction of the Carleman matrix for elliptic systems was carried out by Sh. Yarmukhamedov, N.N. Tarkhanov, A.A. Shlapunov, I.E. Niyozov and others. In papers [9-14] The questions of exact and approximate solutions of the ill-posed Cauchy problem for various factorizations of the Helmholtz equations are studied. Such problems arise in mathematical physics and in various fields of natural science (for example, in electro-geological exploration, in cardiology, in electrodynamics, etc.). Using the construction of previous works, the validity of the fundamental solution for the matrix factorization of the Helmholtz equation in various spaces was proved in the works [15-24]. The solution of the heat equation by solving the integro - differential equation and unusual quantum entanglement consistent with the Schrödinger equation were considered in works [25-27].

Conclusion

Hadamard believed that any mathematical problem corresponding to any physical or technical problem should be correct, since it is difficult to imagine what physical interpretation the solution can have if arbitrarily small changes in the initial data can correspond to large changes in the solution. This called into question the expediency of studying ill-posed problems (examples are given by Hadamard himself). Later it was established that widespread mathematical problems are unstable in certain metrics: the solution of integral equations of the first kind; differentiation of functions known approximately; numerical summation of Fourier series when their coefficient is known approximately; solving systems of linear algebraic equations under conditions of a system determinant close to zero; the Cauchy problem for the Laplace equation; analytic continuation of functions; inverse problems of gravimetry; minimization of functionals; some problems of linear programming and optimal control, as well as optimal design (synthesis of antennas and other physical systems); object control described by differential equations.

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