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On the Cauchy problem for the Helmholtz equation

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Abstract

The work is devoted to the study of continuation and evaluation of the stability of the solution of the Cauchy problem for the Laplace equation in a domain from its known values on the smooth part of the boundary of the domain. The problem under consideration belongs to the problems of mathematical physics, in which there is no continuous dependence of solutions on the initial data. When solving applied problems, it is necessary to find not only an approximate solution, but also a derivative of the approximate solution.

Introduction

The considered problem belongs to ill-posed problems of mathematical physics. In the work of Tikhonov A.N. [1], the true nature of ill-posed problems of mathematical physics was clarified. He pointed out the practical importance of unstable problems and showed that if we restrict the class of possible solutions to a compact set, then the stability of the solution follows from the existence and uniqueness, i.e. the task becomes sustainable. Formulas that make it possible to find a solution to an elliptic equation in the case when the Cauchy data are known only on a part of the boundary of the domain are called Carleman-type formulas. In [2], Carleman established a formula that gives a solution to the Cauchy-Riemann equations in a domain of a special form. Developing his idea, G.M. Goluzin and V.I. Krylov [3] derived a formula for determining the values of analytic functions from data known only on the boundary segment, already for arbitrary regions. They found a formula for recovering a solution from its values on the boundary set of a positive Lebesgue measure, and also proposed a new version of the continuation formula. It is known that the Helmholtz equation in different spaces has a fundamentally different solution. In the future, using the construction of constructing a fundamental solution, we will construct an approximate solution for the Helmholtz equation. The fundamental solution for the Laplace equation was constructed by Sh. Yarmukhamedov [4]. The fundamental solution for the matrix factorization of the Laplace equation was proved in [5]. Using the construction of work [4], as well as work [5], we prove the validity of the fundamental solution for the Helmholtz equation in the plane case. For the matrix factorization of the Helmholtz equation, the validity of fundamental solutions in various spaces was considered by the author [6-18].

Basic information and formulation of the Cauchy problem

This section deals with the construction of a family of fundamental solutions of the Helmholtz equation, parameterized by an entire function with certain properties.

Let R^2 be a two-dimensional real Euclidean space, $\xi=(\xi_1,\xi_2)\in R^2$, $\eta=(\eta_1,\eta_2)\in R^2$, $\alpha=|\eta_1-\xi_1|$, $r=|\eta-\xi|$. $G\subset R^2$ is a bounded simply connected region whose boundary consists of a segment $a\le y_1\le b$ and some smooth curve S lying in the half-plane $y_2>0$, i.e., $\partial G=S\bigcup T$.

We consider the Helmholtz equation in domain G

$$\Delta W(\eta)+\lambda^2 W(\eta)=0, \tag{1}$$

where $\lambda>0$, $\Delta-$ is the Laplace operator.

We denote by $\mathrm{N}(w)$ is an entire function taking real values for real w ($w=u+iv;\ u,v-$real numbers) and satisfying the following conditions:

$$\mathrm{N}(u)\neq 0,\ \sup_{v\geq 1}\left|v^p \mathrm{N}^{(p)}(w)\right|=\mathrm{M}(u,p)<\infty,\ -\infty<u<\infty,\ p=0,1,2. \tag{2}$$

We define a function $\Psi(\eta,x)$ when $\eta\neq\xi$ by the following equality:

$$\Psi(\eta,\xi)=-\frac{1}{2\pi K(\xi_2)}\int_0^\infty \mathrm{Im}\frac{\mathrm{N}(w)}{w-\xi_2}\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du,\ \ w=i\sqrt{u^2+\alpha^2}+\eta_2, \tag{3}$$

where $I_0(\lambda u)-$is the Bessel function of the first kind of zero order.

The function $\Psi(\eta,\xi)$ can be represented as

$$\Psi(\eta,\xi)=-\frac{i}{4}H_0^{(1)}(\lambda r)+\phi(\eta,\xi). \tag{4}$$

Here $-\frac{i}{4}H_0^{(1)}(\lambda r)-$ is the fundamental solution of the Helmholtz equation in R^2, defined through the Hankel function of the first kind, $\phi(\eta,\xi)-$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $\eta=\xi$.

Let $A(G)$ be a set of functions that are solutions of equation (1) in G, continuous with their first-order partial derivatives up to the boundary of ∂G (if the boundary of ∂G extends to infinity, then continuity is required only at the end points of $A(G)$).

The Cauchy problem 1. Suppose $W(\eta)\in A(G)$ and

$$W(\eta)\big|_S = f(\eta),\ \frac{\partial W(\eta)}{\partial n}\bigg|_S = g(\eta),\ \ \eta\in S. \tag{5}$$

Here, $f(\eta)$ and $g(\eta)$ are given continuous vector-function on S.

It is required to restore the vector function $W(\eta)$ in the domain G, based on it's values $f(y)$ on S.

In the formula (3) choosing

$$\mathrm{N}(w)=\exp(\sigma w),\ \mathrm{N}(\xi_2)=\exp(\sigma\xi_2),\ \sigma>0, \tag{6}$$

we get

$$\Psi_\sigma(\eta,\xi)=-\frac{e^{-\sigma\xi_2}}{2\pi}\int_0^\infty \mathrm{Im}\frac{\exp(\sigma w)}{w-\xi_2}\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du,$$
$$\sigma\geq\lambda+\sigma_0,\ \sigma_0>0. \tag{7}$$

For a function $W(\eta)\in A(G)$ and any $\xi\in G$, the following Green's integral formula holds:

$$W(\xi)=\int_{\partial G}\left[\frac{\partial W(\eta)}{\partial n}\Psi_\sigma(\eta,\xi)-f(\eta)\frac{\partial\Psi_\sigma(\eta,\xi)}{\partial n}\right]ds_\eta,\ \ \xi\in G, \tag{8}$$

Theorem 1. Let $W(\eta)\in C^2(G)\bigcap C^1(G)$ it satisfy the inequality

$$|W(\eta)|+\left|\frac{\partial W(\eta)}{\partial n}\right|\leq 1,\ \eta\in T. \tag{8}$$

If

$$W_\sigma(\xi)=\int_S\left[g(\eta)\Psi_\sigma(\eta,\xi)-f(\eta)\frac{\partial\Psi_\sigma(\eta,\xi)}{\partial n}\right]ds_\eta,\ \ \xi\in G, \tag{9}$$

then the following estimate is true

$$|W(\xi)-W_\sigma(\xi)|\leq C(\lambda,\xi)\sigma e^{-\sigma\xi_2},\ \ \sigma>1,\ \ \xi\in G. \tag{10}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C(\lambda,x)$.

Proof. We estimate the integrals $\int_a^b|\Psi_\sigma(\eta,\xi)|ds_\eta$, $\int_a^b\left|\frac{\partial\Psi_\sigma(\eta,\xi)}{\partial\eta_1}\right|ds_\eta$ and $\int_a^b\left|\frac{\partial\Psi_\sigma(\eta,\xi)}{\partial\eta_2}\right|ds_\eta$ on the part T of the plane $y_2=0$.

$$\int_a^b|\Psi_\sigma(\eta,\xi)|ds_y\leq C(\lambda,\xi)\sigma e^{-\sigma\xi_2},\ \ \sigma>1,\ \ \xi\in G. \tag{11}$$

$$\int_a^b \left| \frac{\partial \Psi_\sigma(\eta,\xi)}{\partial \eta_1} \right| ds_\eta \le C(\lambda,\xi)\sigma e^{-\sigma \xi_2}, \ \sigma > 1, \ \xi \in G. \tag{12}$$

$$\int_a^b \left| \frac{\partial \Psi_\sigma(\eta,\xi)}{\partial \eta_2} \right| ds_\eta \le C(\lambda,\xi)\sigma e^{-\sigma \xi_2}, \ \sigma > 1, \ \xi \in G. \tag{13}$$

Combining the estimates of (11), (12) and (13), we obtain the proof of the theorem.

Theorem 1 is proved.

Corollary 1. The limiting equality

$$\lim_{\sigma \to \infty} W_\sigma(\xi) = W(\xi)$$

holds uniformly on each compact set in the domain G.

Conclusion

In the work, using the Carleman function, an unknown function is restored from the Cauchy data on a part of the boundary of the region. If the Carleman function is constructed, then using Green's formula, one can find a regularized solution in an explicit form. It is shown that the efficient construction of the Carleman function is equivalent to the construction of a regularized solution of the Cauchy problem. It is assumed that a solution to the problem exists and is continuously differentiable in a closed domain with exactly given Cauchy data. For this case, an explicit formula for the continuation of the solution and its derivative is established, as well as a regularization formula for the case when, under the specified conditions, instead of the initial Cauchy data, their continuous approximations with a given error in the uniform metric are given. Stability estimates for the solution of the Cauchy problem in the classical sense are obtained.

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