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On the formulation of the Cauchy problem for matrix factorizations of the Helmholtz equation

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Abstract
In this paper, we are talking about the formulation of the Cauchy problem for matrix factorizations of the Helmholtz equation in two-dimensional and three-dimensional bounded domains. Preliminary information and formulation of the Cauchy problem are given.

Introduction

It is known that the Cauchy problem for elliptic equations is incorrect: the solution to the problem is unique, but unstable. The Cauchy problem for matrix factorizations of the Helmholtz equation, like many Cauchy problems for finding regular solutions of elliptic equations, in the general case is unstable with respect to uniformly small changes in the initial data. Thus, these tasks are incorrectly posed [1].

In unstable problems, the image of the operator is not closed, therefore, the solvability condition cannot be written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on a part of the boundary of a domain, the solution is usually unique, the problem is solvable for an everywhere dense data set, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than the theory of solvability of the Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg [2], A.M. Kytmanov and N.N. Tarkhanov [6]. For special domains, the problem of continuing limited analytic functions in the case when data is specified only on a part of the boundary was considered by T. Karleman [7]. The research of T. Karleman was continued by G.M. Goluzin and V.I. Krylov. The use of the classical Green formula for constructing a regularized solution of the Cauchy problem for the Laplace equation was proposed by academician M.M. Lavrent'ev in his famous monograph [4]. Using the ideas of M. M. Lavrent'ev [3-4], Sh. Yarmukhamedov constructed in explicit form a regularized solution of the Cauchy problem for the Laplace equation (see for instance [5]) In work [6], an integral formula was proved for systems of equations of elliptic type of the first order with constant coefficients in a bounded domain.

The construction of the Carleman matrix for elliptic systems was carried out by Sh. Yarmukhamedov, N.N. Tarkhanov, A.A. Shlapunov, I.E. Niyozov and others. In papers [8-9] The questions of exact and approximate solutions of the ill-posed Cauchy problem for various factorizations of the Helmholtz equations are studied. Such problems arise in mathematical physics and in various fields of natural science (for example, in electro-geological exploration, in cardiology, in electrodynamics, etc.)

Basic information and formulation of the Cauchy problem

Let R^2 be a two-dimensional real Euclidean space, $x=(x_1,x_2)\in\mathrm{R}^2$, $y=(y_1,y_2)\in\mathrm{R}^2$.

$G \in \mathrm{R}^2$ is a bounded simply connected domain with a piecewise smooth boundary consisting of the plane $T: y_2 = 0$ and some smooth curve S lying in the half-space $y_2 > 0$, i.e. $\partial G = S \bigcup T$.

We introduce the following notation:

$$r = |y-x|,\ \alpha = |y-x|,\ w = i\sqrt{u^2+\alpha^2} + y_2,\ u \geq 0,\ \partial_x = \left(\partial_{x_1}, \partial_{x_2}\right)^T,\ \partial_x = \xi^T,$$

$$\xi^T = \left(\xi_1\ \xi_2\right)^T - \text{-transposed vektor } \xi,\ U(x) = (U_1(x),...,U_n(x))^T,\ u^0 = (1,...,1) \in \mathrm{R}^n,$$

$$n = 2^m,\ m = 2,\ E(z) = \begin{Vmatrix} z_1 ... 0 \\ .... \\ 0 ... z_n \end{Vmatrix} \text{ - diagonal matrix, } z = (z_1,...,z_n) \in \mathrm{R}^n$$

Let $D(\xi^T)$, $(n \times n)$ - be a matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the condition is satisfied:

$$D^*(\xi^T) D(\xi^T) = E((|\xi|^2 + \lambda^2) u^0) \tag{1}$$

where $D^*(\xi^T)$ - Hermitian conjugate matrix to $D(\xi^T)$, $|\xi|^2 = \sum_{j=1}^{2} \xi_j^2$, $\lambda -$ real number.

Consider in the domain G a system of partial differential equations of the first order with constant coefficients of the form

$$D\left(\partial_x\right) U(x) = 0, \tag{2}$$

where $D\left(\partial_x\right)$ is the matrix of differential operators of the first order.

We denote by $A(G)$ the class of vector functions in the domain G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (2).

The Cauchy problem 1. Suppose $U(y) \in A(G)$ and

$$U(y)\big|_S = f(y),\ y \in S. \tag{3}$$

Here, $f(y)$ a given continuous vector-function on S.

It is required to restore the vector function $U(y)$ in the domain G, based on it's values $f(y)$ on S.

Example 1. Let given a system of first-order partial differential equations of the form

$$\begin{cases} \partial_{x_1} U_1 - \partial_{x_2} U_2 + i U_4 = 0, \\ \partial_{x_2} U_1 + \partial_{x_1} U_2 + i U_3 = 0, \\ -\partial_{x_1} U_3 + \partial_{x_1} U_4 - i U_2 = 0, \\ \partial_{x_2} U_3 + \partial_{x_1} U_4 + i U_1 = 0. \end{cases}$$

Assuming $\partial_{x_1} \to \xi_1, \partial_{x_2} \to \xi_2$, we compose the following matrices:

$$D(\xi^T) = \begin{pmatrix} \xi_1 & \xi_2 & 0 & i \\ -\xi_2 & \xi_1 & -i & 0 \\ 0 & i & -\xi_1 & \xi_2 \\ i & 0 & \xi_2 & \xi_1 \end{pmatrix},\quad D^*(\xi^T) = \begin{pmatrix} \xi_1 & -\xi_2 & 0 & -i \\ \xi_2 & \xi_1 & -i & 0 \\ 0 & i & -\xi_1 & \xi_2 \\ -i & 0 & \xi_2 & \xi_1 \end{pmatrix}.$$

Relationship (1) is easily verified.

Let R^3 be the three-dimensional real Euclidean space, $x = (x_1, x_2, x_3) \in \mathrm{R}^3$, $y = (y_1, y_2, y_3) \in \mathrm{R}^3$, $x' = (x_1, x_2) \in \mathrm{R}^2$, $y' = (y_1, y_2) \in \mathrm{R}^2$.

$G \subset \mathrm{R}^3$ be a bounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_3 = 0$ and of a smooth surface S lying in the half-space $y_3 > 0$, that i.s., $\partial G = S \bigcup T$.

We introduce the following notation:

$$r = |y-x|, \alpha = |y'-x'|,\ w = i\sqrt{u^2+\alpha^2} + y_3, u \geq 0,\ \partial_x = \left(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}\right)^T,\ \partial_x \to \xi^T,$$

$$\xi^T = (\xi_1\ \xi_2\ \xi_3)^T - \text{ transposed vector } \xi,\ U(x) = (U_1(x),...,U_n(x))^T,\ u^0 = (1,...,1) \in \mathrm{R}^n,$$

$$n = 2^m,\ m = 3,\ \ E(z) = \begin{Vmatrix} z_1 \ldots 0 \\ \ldots\ldots. \\ 0 \ldots z_n \end{Vmatrix} - \text{ diagonal matrix, } z = (z_1,...,z_n) \in \mathrm{R}^n.$$

Let $D(x^T)$ the $(n \times n)$ – the matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0), \tag{4}$$

where $D^*(x^T)$ is the Hermitian conjugate matrix $D(x^T)$, $|\xi|^2 = \sum_j^3 \xi_j^2$, λ – real number

Consider in the region G a system of differential equations in partial derivatives of the first order

$$D(\partial_x)U(x) = 0, \tag{5}$$

where $D(\partial_x)$ is the matrix of differential operators of the first order.

We denote by $A(G)$ the class of vector functions in the domain G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (5).

The Cauchy problem 2. Suppose $U(y) \in A(G)$ and

$$U(y)|_S = f(y),\ y \in S. \tag{6}$$

Here, $f(y)$ a given continuous vector-function on S.

It is required to restore the vector function $U(y)$ in the domain G, based on it's values $f(y)$ on S.

Example 2. Let a system of first-order partial differential equations of the form

$$\begin{cases} \partial_{x_1}U_1 + \partial_{x_2}U_4 + \partial_{x_3}U_6 + iU_8 = 0, \\ \partial_{x_1}U_2 + \partial_{x_2}U_3 + \partial_{x_3}U_5 + iU_7 = 0, \\ \partial_{x_2}U_2 - \partial_{x_1}U_3 + \partial_{x_3}U_8 + iU_6 = 0, \\ -\partial_{x_2}U_1 + \partial_{x_1}U_4 + \partial_{x_3}U_7 + iU_5 = 0, \\ \partial_{x_3}U_2 + \partial_{x_1}U_5 + \partial_{x_2}U_8 + iU_4 = 0, \\ \partial_{x_3}U_1 - \partial_{x_1}U_6 + \partial_{x_2}U_7 + iU_3 = 0, \\ \partial_{x_3}U_4 - \partial_{x_2}U_6 + \partial_{x_3}U_7 + iU_2 = 0, \\ \partial_{x_3}U_3 + \partial_{x_2}U_5 + \partial_{x_1}U_8 + iU_1 = 0. \end{cases}$$

Assuming $\partial_{x_1} \to \xi_1$, $\partial_{x_2} \to \xi_2$ and $\partial_{x_3} \to \xi_3$, we obtain the matrices

$$D(\xi^T) = \begin{pmatrix} \xi_1 & 0 & 0 & \xi_2 & 0 & \xi_3 & 0 & i \\ 0 & \xi_1 & \xi_2 & 0 & \xi_3 & 0 & i & 0 \\ 0 & \xi_2 & -\xi_1 & 0 & 0 & i & 0 & \xi_3 \\ -\xi_2 & 0 & 0 & \xi_1 & i & 0 & \xi_3 & 0 \\ 0 & \xi_3 & 0 & i & \xi_1 & 0 & 0 & \xi_2 \\ \xi_3 & 0 & i & 0 & 0 & -\xi_1 & \xi_2 & 0 \\ 0 & i & 0 & \xi_3 & 0 & -\xi_2 & \xi_1 & 0 \\ i & 0 & \xi_3 & 0 & \xi_2 & 0 & 0 & \xi_1 \end{pmatrix},\ D^*(\xi^T) = \begin{pmatrix} \xi_1 & 0 & 0 & -\xi_2 & 0 & \xi_3 & 0 & -i \\ 0 & \xi_1 & \xi_2 & 0 & \xi_3 & 0 & -i & 0 \\ 0 & \xi_2 & -\xi_1 & 0 & 0 & -i & 0 & \xi_3 \\ \xi_2 & 0 & 0 & \xi_1 & -i & 0 & \xi_3 & 0 \\ 0 & \xi_3 & 0 & -i & \xi_1 & 0 & 0 & \xi_2 \\ \xi_3 & 0 & -i & 0 & 0 & -\xi_1 & -\xi_2 & 0 \\ 0 & -i & 0 & \xi_3 & 0 & \xi_2 & \xi_1 & 0 \\ -i & 0 & \xi_3 & 0 & \xi_2 & 0 & 0 & \xi_1 \end{pmatrix}$$

Relation (4) is easily verified.

Conclusion

In this paper, we present the basic concepts and formulation of the Cauchy problem for matrix factorizations of the Helmholtz equation. To prove the conditions for the matrix factorization of the Helmholtz equation to be satisfied, the corresponding examples are given. On the basis of these examples, approximate solutions of the Cauchy problem for matrix factorizations of the Helmholtz equation are found (see, for instance [8-9]).

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