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Fundamental solution for the Helmholtz equation in the plane

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Abstract

This paper deals with the construction of a family of fundamental solutions of the Helmholtz equation, parameterized by an entire function with certain properties. The lemma for the Helmholtz equation on a two-dimensional bounded domain is proved.

Introduction

It is known that the Helmholtz equation in different spaces has a fundamentally different solution. In the future, using the construction of constructing a fundamental solution, we will construct an approximate solution for the Helmholtz equation. The fundamental solution for the Laplace equation was constructed by Sh. Yarmukhamedov [1]. The fundamental solution for the matrix factorization of the Laplace equation was proved in [2]. Using the construction of work [1], as well as work [2], we prove the validity of the fundamental solution for the Helmholtz equation in the plane case.

For the matrix factorization of the Helmholtz equation, the validity of fundamental solutions in various spaces was considered by the author [3-9].

Basic information and formulation of the Cauchy problem

This section deals with the construction of a family of fundamental solutions of the Helmholtz equation, parameterized by an entire function with certain properties.

Let R^2 be a two-dimensional real Euclidean space,

$$x=(x_1,x_2)\in R^2,\ y=(y_1,y_2)\in R^2,\ \alpha=|y_1-x_1|,\ r=|y-x|.$$

$G\subset R^2$ is a bounded simply connected region whose boundary consists of a smooth curve $S=\partial G,\ \overline{G}=S\bigcup G$. We consider the Helmholtz equation

$$\Delta U(y)+\lambda^2 U(y)=0, \tag{1}$$

where $\lambda>0$, $\Delta-$ is the Laplace operator.

We denote by $K(w)$ is an entire function taking real values for real w ($w=u+iv; u,v-$ real numbers) and satisfying the following conditions:

$$K(u)\neq 0,\ \sup_{v\geq 1}\left|v^p K^{(p)}(w)\right|=\mathrm{M}(u,p)<\infty,\ -\infty<u<\infty,\ p=0,1,2. \tag{2}$$

We define a function $\Phi(y,x)$ when $y\neq x$ by the following equality:

$$\Phi(y,x)=-\frac{1}{2\pi K(x_2)}\int_0^\infty \mathrm{Im}\frac{K(w)}{w-x_2}\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du,\ \ w=i\sqrt{u^2+\alpha^2}+y_2, \tag{3}$$

where $I_0(\lambda u)$ – is the Bessel function of the first kind of zero order.

Lemma. *The function* $\Phi(y,x)$ *can be represented as*

$$\Phi(y,x) = -\frac{i}{4}H_0^{(1)}(\lambda r) + g(y,x). \tag{4}$$

Here $-\frac{i}{4}H_0^{(1)}(\lambda r)-$ is the fundamental solution of the Helmholtz equation in P^2, defined through the Hankel function of the first kind, $g(y,x)-$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

We note that the proof of the lemma remains valid if, in (3), for $K(w)$ we take an analytic function that is regular in some domain and takes real values for real w, satisfying condition (2).

Proof. For convenience, we introduce the notation

$$f(w) = \frac{K(w)}{w - x_2}, \quad w = i\sqrt{u^2+\alpha^2} + y_2, \quad \varphi(y,x) = \int_0^\infty f(w)\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du. \tag{5}$$

In these notation

$$\Phi(y,x) = \frac{1}{2\pi K(x_2)}\operatorname{Im}\varphi(y,x). \tag{6}$$

Our goal is to prove that the function $\varphi(y,x)$ is a solution of equation (1) with respect to the variable y at $\alpha > 0$.

This will follow that the function $\Phi(y,x)$ is a solution of equation (1) with respect to y at $\alpha > 0$.

Taking into account conditions (2), from formula (5), by differentiation we obtain

$$\frac{\partial \varphi(y,x)}{\partial y_1} = \int_0^\infty \frac{\alpha u i f'(w)}{u^2+\alpha^2}I_0(\lambda u)du - \int_0^\infty \frac{\alpha u f(w)}{\left(u^2+\alpha^2\right)^{3/2}}I_0(\lambda u)du, \quad y_1 > x_1. \tag{7}$$

The first integral is integrable by parts

$$\int_0^\infty \frac{\alpha u i f'(w)}{u^2+\alpha^2}I_0(\lambda u)du = \int_0^\infty \frac{\alpha I_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du =$$

$$= -f(w) + \int_0^\infty \frac{\alpha u I_0(\lambda u)}{\left(u^2+\alpha^2\right)^{3/2}}f(w)du - \lambda\int_0^\infty \frac{\alpha I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f(w)du.$$

Substituting these expressions in (7), we obtain

$$\frac{\partial \varphi(y,x)}{\partial y_1} = -f(w) - \lambda\int_0^\infty \frac{\alpha I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f(w)du, \quad y_1 > x_1. \tag{8}$$

In the same way, we get

$$\frac{\partial \varphi(y,x)}{\partial y_1} = f(w) - \lambda\int_0^\infty \frac{\alpha I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f(w)du, \quad y_1 < x_1. \tag{9}$$

Taking into account (8) and (9), we have

$$\frac{\partial \varphi^2(y,x)}{\partial y_1^2} = -if(w) - \lambda\int_0^\infty \frac{I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f(w)du +$$

$$+\int_0^\infty \frac{\alpha^2 I_0'(\lambda u)}{\left(u^2+\alpha^2\right)^{3/2}}f(w)du - \lambda\int_0^\infty \frac{i\alpha^2 I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f'(w)du, \quad y_1 \neq x_1,$$

or

$$\frac{\partial \varphi^2(y,x)}{\partial y_1^2} = -if'(w) - \lambda\int_0^\infty \frac{u^2 I_0'(\lambda u)}{\left(u^2+\alpha^2\right)^{3/2}}f(w)du - \int_0^\infty \frac{i\alpha^2 I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f'(w)du. \tag{10}$$

Now we will calculate the partial derivatives of the function $\varphi(y,x)$ with respect to y_2 at $y_1 \neq x_1$.

$$\frac{\varphi(y,x)}{\partial y_2} = \int_0^\infty \frac{f'(w)\,u\,I_0(\lambda u)}{\sqrt{u^2+\alpha^2}} f(w)du, \quad y_1 \neq x_1.$$

Integrating in parts, we obtain

$$\frac{\partial\varphi(y,x)}{\partial y_2} = f(w) + i\lambda\int_0^\infty I_0'(\lambda u)\,f(w)du, \;\; y_1 \neq x_1. \tag{11}$$

From here

$$\frac{\partial^2\varphi(y,x)}{\partial y_2^2} = if'(w) + i\lambda\int_0^\infty I_0'(\lambda u)\,f'(w)\,du, \;\; y_1 \neq x_1. \tag{12}$$

Taking into account (10), (12) and (1), we have

$$\Delta\varphi(y,x) + \lambda^2\varphi(y,x) = -\lambda\int_0^\infty \frac{u^2 I_0'(\lambda u)}{\left(u^2+\alpha^2\right)^{3/2}} f(w)du +$$

$$+i\lambda\int_0^\infty \frac{u^2 f'(w)\,I_0'(\lambda u)}{u^2+\alpha^2}du + \lambda^2\int_0^\infty \frac{f(w)\,I_0'(\lambda u)u}{\sqrt{u^2+\alpha^2}}du, \quad y_1 \neq x_1.$$

Integrating the second integral in parts, we obtain

$$\Delta\varphi(y,x) + \lambda^2\varphi(y,x) = -\lambda\int_0^\infty \frac{f(w)}{\sqrt{u^2+\alpha^2}}\left[\lambda u I_0''(\lambda u) + I_0'(\lambda u) + \lambda u I_0(\lambda u)\right]du.$$

Since, the integrand

$$\lambda u I_0''(\lambda u) + I_0'(\lambda u) + \lambda u I_0(\lambda u) = 0$$

is the zero order Bessel equation and $I_0(\lambda u)$ is its solution, then

$$\Delta\varphi(y,x) + \lambda^2\varphi(y,x) = 0, \;\; y_1 \neq x_1.$$

It follows from this equality that $\Phi(y,x)$ is a solution of equation (1) with respect to y on the line $y_1 \neq x_1$. For this it is enough to show its differentiability as $y_1 = x_1$ (then, according to the well-known property of solving an elliptic equation, it continues on the line $y_1 = x_1$ as a solution).

Similarly, take the partial derivative of the function $\Phi(y,x)$ with respect to y_2, and we can completely prove the lemma. But due to the limitation of the number of pages, we will give a complete proof of the lemma in the future in other papers.

The lemma proved.

Conclusion

In this work, on the basis of previous research works, we proved the validity of the fundamental solutions of the Helmholtz equation. The construction of a fundamental solution allows, in the future, to find in an explicit form a regularized solution for the Helmholtz equation.

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