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History of ill-posed problems and their application to solve various mathematical problems

Davron Aslonqulovich Juraev [*1,4], **Ebrahim Eldesoky Elsayed** [2], **Juan Jose Diaz Bulnes** [3], **Praveen Agarwal** [4], **Rostam Karim Saeed** [5]

[1] *University of Economics and Pedagogy, Department of Scientific Research, Innovation and Training of Scientific and Pedagogical Staff, Uzbekistan, juraevdavron12@gmail.com*
[2] *Mansoura University, Department of Electronics and Communications Engineering, Egypt, engebrahem16@gmail.com*
[3] *Federal University of Amapá, Department of Exact Sciences and Technology, Brazil, bulnes@unifap.br*
[4] *Anand International College of Engineering, Department of Mathematics, India, goyal.praveen2011@gmail.com*
[5] *Salahaddin University, College of Science, Iraq, rostam.saeed@su.edu.krd*

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Abstract

This study aims to provide an understanding of well-posed and incorrectly-posed problems, as well as the developed methods for solving incorrectly-posed applied problems in mathematics. The history and significance of incorrectly-posed problems in solving various applied problems in the natural sciences are explored in detail. The study of methods for solving ill-posed problems has garnered significant interest among researchers, who are actively conducting research in this field. The theory of incorrectly-posed problems is a rapidly developing area in mathematical physics and natural sciences.In the practical realm, most problems are ill-posed, requiring decision-making under conditions of uncertainty, overdetermination, or inconsistency. The main conclusion drawn from this study is that solving incorrectly-posed problems cannot be accomplished solely by learning from well-posed problems.

1. Introduction

The purpose of this study is to explain the theory of correct and incorrectly posed problems, as well as to present the methods developed for solving incorrect problems that arise in applied mathematics. The origins of ill-posed problems, their significance, and their role in resolving a variety of applied problems in natural science are elaborated upon in detail. Currently, the exploration of methods for solving incorrectly posed challenges has piqued the interest of researchers worldwide, stimulating an active wave of research. The theory of incorrectly posed problems is presently a rapidly developing field in mathematical physics and natural sciences. In reality, most practical problems are ill-posed, necessitating decision-making under uncertainty, over determination, or inconsistency. The key takeaway is that one cannot learn to tackle ill-posed problems solely from approaching well-posed ones.

A multitude of applied problems such as geophysical, biophysical, electrodynamics, aerodynamics, plasma physics can be simplified to equations of mathematical physics. The construction of these equations, which adequately narrate specific physical laws, is a solution to an 'inverse' problem. The investigator observes a phenomenon and constructs equations whose solutions carry similar properties. The result of this construing process is typically based on general laws that enable the formulation of differential relationships. These relationships usually consist of arbitrary functions representing the properties of the physical medium under study.

Hadamard's definition of problem correctness in mathematical physics has been detailed above. When one of the conditions 1-3 is not fulfilled, these problems are classified as ill-posed. The Carleman functions, constructed in 1926 by Carleman [1], have been instrumental in formulating regularizations and estimating the conditional stability of mathematical challenges.

Significantly, the consideration of ill-posed problems in mathematical physics was first deemed necessary by Tikhonov [2-4] while interpreting geophysical observational data. Tikhonov's work showed that reducing a problem's number of possible solutions to a compact set generates stability from the existence and uniqueness of the solution [2-4]. Prominent scientists have been credited with further developments in the theory and applications of ill-posed problems [2-7].

Upon understanding the physics, these mathematical equations combined with initial and boundary conditions can predict the development of a physical phenomenon in space-time. These are known as 'direct' problems. In modern science, inverse problems arise when the general form of these equations is known, but the characteristic properties of the medium are unknown, thus requiring derivation from observed equation solutions. In mathematical terms, these problems must guarantee the correctness of the problem statement.

The concept of problem correctness for mathematical physics equations was first introduced in the work of Hadamard [8]. A problem is considered well-posed if it satisfies three criteria: 1) the problem must have a solution, 2) the solution must be unique, and 3) the solution must be stable. After establishing the term for correctness in mathematical physics problems, J Hadamard gave an example of an ill-posed problem that didn't align with any physical formulation of the problem. He demonstrated this with the Cauchy problem for the Laplace equation, now known as the classic example of an ill-posed problem.

The stability requirement in mathematical physics problems implies that any physically defined process must consistently depend on the initial and boundary conditions. When problem solutions are sensitive to slight changes in the initial data, such problems are deemed as ill-posed. These tend to cause large variations in solutions from small changes in the initial data.

2. Evolution of Ill-Posed Problems in Current Mathematical Sciences

The Cauchy problem for the Laplace equation is considered in papers [6-8]. Renowned mathematician Yarmukhamedov [9] proposed a method to construct a family of fundamental solutions. This method forms explicit formulas to recover solutions of elliptic problems from their Cauchy data on a section of the boundary. Formulas that have such properties as known are called formulas or Carleman matrices [9-12].

In solving mathematical physics problems, the operator's image is not closed, making them unsolvable in terms of continuous linear functions. Such issues make the theory of solvability for these problems complex and profound. Multiple studies have explored the nature and properties of these problems, along with the numerical solutions for some mathematical physics problems. Based on earlier research, the regularization of the Cauchy problem for the Helmholtz equation was accomplished. Specific equations' boundary problems and some mathematical physics problems' numerical solutions have been considered.

The Cauchy problem for most elliptic equations has a unique solution, i.e., this problem is solvable for an everywhere dense data set of the problem, but this data set is not considered closed. It follows that the theory of solvability of similar problems are very complex. In [13-21] one can learn in detail about the nature and properties of such problems. Approximate solutions of the ill-posed Cauchy problem for different factorizations of the Helmholtz operator are considered in [22–32]. Based on these results, a regularized solution of the Cauchy problem was found explicitly for various factorizations of the Helmholtz operator [33–49]. To distinguish between correct and incorrect problems, readers are provided with the following works, which contain sufficiently necessary information. For example, in works [50-51], integro-differential equations are considered, in which numerical results are obtained. Similar methods for solving problems for the Schrödinger's equation, as well as for integral equations, can be found in works [52-61]. The Ulam-Hayers-Rassias stability of some quasilinear first-order partial differential equations is considered in papers [62-66]. The inverse problem of determining the source function in the Riemann-Liouville fractional derivative equation was studied in a joint paper [67]. Nurieva [68] considered the problem of constructing two-way multistep methods and their application to solve the Volterra integral equation. Islomov and Kholbekov [69-70] solved a boundary value problem for an equation of parabolic-hyperbolic type loaded with an integral operator of fractional order. Boundary value problems for various equations of mathematical physics, as well as for some modified equations, were considered in papers [71-76]. Muhammad, et al. [77] found a new algorithm for computing Adomian polynomials for solving the coupled Hirota system. Solving the Volterra-Fredholm integral equations using a natural cubic spline function was considered by Salim, et al. [78].

In works [79-80], the regularization of the Cauchy problem for Helmholtz equations, as well as for elliptic equations with constant coefficients, was considered.

3. Examples of Ill-Posed Problems

We'll provide various examples that illustrate the unpredictability associated with ill-posed problems [81]. Consider an operator equation of the first category as given in Equation 1.

$$Dv = g \tag{1}$$

in which D is a given mapping (operator) from a topographical space V into another topographical space B. Our objective is to discover $v \in V$ from a specified B. It's widely accepted that the $\{D, g\}$ data we're privy to are approximations at best.

Example 1. Assume that in system (1), D is a matrix with dimension $m \times n$. Here are the scenarios to be considered:

1) If $m > n$, It's deducible that a solution to Equation (1) exists, and it could potentially not be singular.
2) If $m < n$, then Equation (1) lacks a solution.
3) If $m = n$ and $\det D \neq 0$, we assume that system (1) is solvable regardless of the right term. The inference here is that the reverse operator D^{-1} (a matrix) exists, meaning it's capped. This indicates the full correctness condition is met.

Now, consider how the problem's solution is dependent upon the fluctuation of the f vector in the event the D matrix isn't singular.

Let us detract Equation (1) from the adjusted Equation 2.

$$D(v + \sigma v) = g + \sigma g \tag{2}$$

This gives us Equation 3.

$$D\sigma v = \sigma g \tag{3}$$

This results in Equation 4.

$$\sigma v = D^{-1}\sigma g, \; \|\sigma v\| = \|D^{-1}\|\|\sigma g\| \tag{4}$$

However, on the converse side (Equation 5):

$$\|D\|\|v\| \geq \|g\| \tag{5}$$

From these inferred relations, an estimation surface (Equation 6):

$$\frac{\|\sigma v\|}{\|v\|} \leq \|D\|\|D^{-1}\|\frac{\|\sigma g\|}{\|g\|}. \tag{6}$$

This suggests the error is reliant upon the exact constant $\mu(D) = \|D\|\|D^{-1}\|$. We note that for a normalized D matrix, namely when condition $\|D\| = 1$ is confirmed, the reverse matrix D^{-1} - has large entries. This signifies that even minimal changes in D data can trigger significant solution changes to the problem.

This suggests that a system with an ill-conditioned matrix should likely be deemed pragmatically unstable, even if the problem is deemed correctly posed and condition $\|D^{-1}\| < \infty$ is met.

Consider the following matrix (Equation 7):

$$\begin{Vmatrix} 1 & d & 0 & \ldots & 0 \\ 0 & 1 & d & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & d \\ 0 & 0 & 0 & \ldots & 1 \end{Vmatrix} \tag{7}$$

This matrix (Equation 7) is ill-conditioned for large n and $|d|>1$. Therefore, the reverse matrix includes components such as d^{n-1}.

For the adjusted matrix, estimate (Equation 6) appears as Equation 8.

$$\frac{\|\sigma v\|}{\|v\|} \le \mu(D)\frac{\frac{\sigma D}{\|D\|}}{1-\mu(D)\frac{\|\sigma D\|}{\|D\|}} \tag{8}$$

(where $\|D^{-1}\|\|\sigma D\|<1$). If D is assumed to be a symmetric matrix and its norm in tandem with the Euclidean norm, then (Equation 9).

$$\mu(D)=|\gamma_{\max}|/|\gamma_{\min}| \tag{9}$$

where maxmin $\gamma_{\max}(\gamma_{\min})$ denote the maximum and minimum eigenvalue of the matrix D.

Example 2. We hypothesize that Ω is a sealed region on a plane, (ξ,τ) (Equation 10):

$$\Omega=\{0\le\xi\le\pi,\ 0\le\tau\le\alpha\pi\}, \tag{10}$$

it's assumed that α holds positive value. Further, the function $v(\xi,t)\in C^2(\Omega)$ is called the solution of the Dirichlet problem or, speaking under another α -wave equation, if the following conditions are met (Equations 11-13):

$$\frac{\partial^2 v}{\partial\tau^2}-\frac{\partial^2 v}{\partial\xi^2}=0,\ (\text{in }\Omega), \tag{11}$$

$$v(0,\tau)=0,\ v(\pi,\tau)=0,\ (0\le\tau\le\alpha\pi), \tag{12}$$

$$v(\xi,0)=\varphi(\xi),\ v(\xi,\alpha\pi)=\psi(\xi),\ (0\le\xi\le\pi). \tag{13}$$

Here, the functions $\varphi(\xi),\ \psi(\xi)$ are continuous over the interval $0\le\xi\le\pi$. We present separately that the solution of problem (11)-(13) is considered to be continuously independent of the initial data $\{\varphi,\psi,\alpha\}$. It can be seen that if α takes an irrational value, then $\varphi_n=0,\ \psi_n=(1/\sqrt{n})\sin\xi$. Using the method of separation of variables, we obtain Equation 14.

$$v_n(\xi,\tau)=\frac{1}{\sqrt{n}}\frac{\sin n\tau\sin n\xi}{\sin n\pi\alpha}. \tag{14}$$

Then, for a sequence of integers $\mu_n,\ \vartheta_n$, the following inequality (Equation 15) is true:

$$|\alpha-\mu_n/\vartheta_n|<1/\vartheta_n^2. \tag{15}$$

Then (Equation 16)

$$|\sin\vartheta_n\alpha\pi|=|\sin(\vartheta_n\alpha-\mu_n)\pi|<\pi/\vartheta_n, \tag{16}$$

whence (Equation 17-18)

$$\sup_{\xi,\tau\in\Omega}|v_{\vartheta_n}(\xi,\tau)|>\sup_{\xi,\tau\in\Omega}\frac{\sqrt{\vartheta_n}}{\pi}|\sin\vartheta_n v\sin\vartheta_n\xi|\to\infty,\text{at}n\to\infty, \tag{17}$$

$$\varphi_n = 0, \psi_n = \frac{1}{\sqrt{\vartheta_n}} \sin \vartheta_n \xi \rightrightarrows 0, \text{at} n \to \infty. \tag{18}$$

Moreover, that first two Hadamard's conditions are not fulfilled, either. For examle, for a rational α the solution is not unique and exists only if the functions $\varphi(\xi)$ an $\psi(\xi)$ are connected by some relations.

Example 3. We consider the classical problem posed by J.Hadamard. It is required to find the solution $v(\xi,\eta)$ of the Laplace Equation 19:

$$\Delta v \equiv v_{\xi\xi} + v_{\eta\eta} = 0 \tag{19}$$

in the domain $\Xi = \{(\xi,\eta) \in E^2 : \eta > 0\}$ satisfies the conditions (Equation 20):

$$v(\xi,0) = 0, \ v_\eta(\xi,0) = g(\xi) = A_n \sin n\xi; \ A_n \to 0 \text{ as } n \to \infty \tag{20}$$

Here, under the Cauchy problem is meant the solution of the Equation (19), which satisfies the condition (Equation 20). The solution is provided by Equation 21.

$$v(\xi,\mu) = \frac{A_n}{n} \sin n\xi \, \text{sh}\, n\eta, \tag{21}$$

which, if $A_n = \frac{1}{n}$, becomes very large for $\eta > 0$, since $\text{sh}\, n\eta = 0e^{n\eta}$.

As $n \to \infty$, the Cauchy data approaches zero in $C'(\mathrm{R})$, and $v \equiv 0$ equals the solution to the equation (*) with $v = v_\eta = 0$. Since the Cauchy problem (19)-(21), although it has a unique solution, it does not depend continuously on the data. From here we are convinced that the Cauchy problem for the Laplace equation (*) is set incorrectly.

Example 4. Consider the issue of numerical differentiation, an incorrectly posed problem since small fluctuations in a differentiable function can lead to major errors in the derivative. Assume $g \in C'[0,1]$ with noisy data $\{\sigma, g_\sigma\}$, where $\sigma > 0$ is the level of noise, we have the estimate (Equation 22).

$$\|g_\sigma - g\| \le \sigma \tag{22}$$

The goal here is to reliably estimate the derivative g', to locate an operation R_σ to estimate the following error (Equation 23).

$$\|R_\sigma g_\sigma - g'\| \le \eta(\sigma) \to 0, \ \sigma > 0 \tag{23}$$

This is synonymous to a stable solution for the Equation 24.

$$Dv := \int_0^\xi v(t)dt = g(\xi), \ D : H := L^2[0,1] \to L^2[0,1]; \ g(0) = 0 \tag{24}$$

where ξ is the independent variable and t its derivative.

When data g_σ, that may contain some noise, is provided instead of g', finding $g' = D^{-1}g$ from the data g_σ can potentially be problematic, as Equation (24) may not have a solution in $L^2[0,1]$ if $g_\sigma \in L^2[0,1]$ is arbitrarily provided, taking into account only limitations of $\|g_\sigma - g\| \le \sigma$, and if $g_\sigma \in C'[0,1]$. Then, g'_σ may significantly diverge from g', irrespective of how small σ is.

Task: For given $\{\sigma, D, g_\sigma\}$, find a stable approximation v_σ to the solution $v(\xi) = g'(\xi)$ of Equation (24) that produces an estimation of the error (Equation 25).

$$\|v_\sigma - v\| \le \eta(\sigma) \to 0 \text{ as } \sigma \to 0 \tag{25}$$

Here, we construct an operator: $R_\alpha : H \to H$ such that (Equation 26).

$$v_\sigma := R_{\alpha(\sigma)} g_\sigma \tag{26}$$

satisfying (Equation 25). Here, R_α depends on parameter α and is termed a regularizer when applied to any $g_\sigma \in Y$.

Example 5. We'll consider a quadratic polynomial of the form (Equation 27):

$$\xi^2 + 2\xi + 1 \tag{27}$$

This has roots $\xi_1 = \xi_2 = -1$ - easily computed with the quadratic formula (Equation 28):

$$\xi_{1,2} = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c} = -1 \pm \sqrt{1-1} = -1 \tag{28}$$

But, if the last term's coefficient is not exactly one (e.g., $1+\varepsilon$, where ε can be any small discrepancy in the coefficient), the real solution vanishes for any small $\varepsilon > 0$, thus making multiple real root-finding a challenging task (Equation 29).

$$\xi_{1,2} = -1 \pm \sqrt{1-(1+\varepsilon)} = -1 \pm \sqrt{\varepsilon} \tag{29}$$

This implies that the problem of finding multiple real roots is considered incorrect.

Example 6. Consider the Equation 30:

$$\frac{d^2\xi}{d\tau^2} - \xi = 0 \tag{30}$$

with boundary conditions $\xi(\tau = 0) = 0,\ \xi(\tau = a) = b$.
Its solution is given by Equation 31.

$$\xi = c_1 \sin\tau + c_2 \cos\tau \tag{31}$$

where c_1 and c_2 constants that are the limits of integration (Here the functions $c_1 \sin\tau$ and $c_2 \cos\tau$ satisfy this equation and their linear combinations correspond to Equation 31.

Taking into account the first boundary condition, we obtain $c_2 = 0$, and from the second $\xi(\tau = a) = b$, respectively, we obtain Equation 32.

$$c_1 = \frac{b}{\sin a} \tag{32}$$

Therefore, we get the following (Equation 33):

$$\xi_1(\tau) = b\frac{\sin\tau}{\sin a} \tag{33}$$

Similarly, if instead of the point $\tau = a + \varepsilon$ we supply $\tau = a$, we have (Equation 34):

$$\xi_2(\tau) = b\frac{\sin\tau}{\sin(a+\varepsilon)} \tag{34}$$

And when $a = \pi - \varepsilon$, then the absolute difference between $\xi_1(\tau)$ and $\xi_2(\tau)$ will be large for small ε.

Let's look at another interesting problem: Determining a ship's position from bearing data. When the shore is in sight, the navigator measures the angles between north and two known beacons A and B. The navigator then plots these bearings on a map. The ship's position can be determined as the intersection point of the two bearings. We therefore need to solve the systems graphically (see, for example, Figure 1) (Equation 35):

$$\begin{cases} d_{11}x + d_{12}y = c_1, \\ d_{21}x + d_{22}y = c_2. \end{cases} \tag{35}$$

It is fundamental that the angles were measured with a certain precision, and errors in drawing lines on the map are considered inevitable; thus, we can only solve the position problem with an estimated accuracy, which is satisfactory for us.

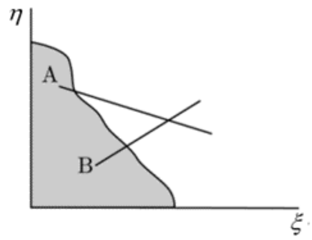

Figure 1. Graphical solution of the system.

Consider the scenario when two beacons are closely positioned, their directions are practically identical, and the drawn lines intersect at a particularly sharp angle, basically converging for certain maps (see, for example, Figure 2).

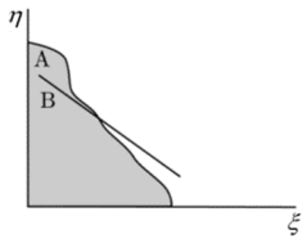

Figure 2. Lines intersecting at an acute angle.

Let's delve into an instance when we aim to solve an incorrectly posed problem, specifically, to solve a system of equations where the determinant of the system is assumed to be zero. Precisely, this problem bores down to either of the following Equation 36.

$$\begin{cases} \xi + \eta = 1, \\ \xi + \eta = 1. \end{cases} \tag{36}$$

In this problem, we can't definitively determine the ship's position from the two coinciding straight lines from the equations $\xi + \eta = 1$ and $\xi + \eta = 1$, it could be anywhere along the line $\xi + \eta = 1$.

We infer that an incorrectly posed problem can be regularized. Consider the regularization method where we aim to find the Equation (35) that has the minimal norm. Assume one of the possible norms is the square root of the sum of ξ and η squares. The problem to solve is to find ξ and η that minimize the square of the given norm $G = \xi^2 + \eta^2$ considering Equations (35) with $\xi + \eta = 1$.

Noting $\xi = 1 - \eta$, we can simplify $G = (1-\eta)^2 + \eta^2$. On considering the derivative of $\frac{\partial G}{\partial \eta}$, it becomes clear that G will be minimal at the point $\xi = 0{,}5$ and $\eta = 0{,}5$. Therefore, from $G = \xi^2 + \eta^2$ we move from an incorrectly posed problem to a well-posed one. But in the practical scenario of navigating a ship's position, this minimal point solution doesn't provide an accurate position for the ship. If the method fails to yield a solution, an alternative optimal method should be sought.

Consider employing supplementary information: if the distances to the matching beacons could be estimated, these data could be used as additional information. Using a compass, the accurate ship's position on the line

$\xi+\eta=1$ can be determined. Another method employs supplementary bearing information from a third, distant beacon. The intersection of this new azimuth with the previous line $\xi+\eta=1$, gives the desired ship's position.

Therefore, using the supplementary information method can serve as a proficient regularization method when the position problem is ill-posed. In this case, introducing the norm and seeking a solution which minimizes the norm, the sequence of well-posed problems tends to the ill-posed problem in the limit.

There happens to be a variety of methods for handling incorrectly posed problems. Let's look at a specific example where we can apply some general theorems, such as finding roots for a polynomial of any degree (Equation 37).

$$d_n\gamma^n + d_{n-1}\gamma^{n-1} + \ldots + d_0. \tag{37}$$

The study presented in this work sheds light on ill-posed problems in the context of solving applied physical problems in mathematical physics. In mathematical language, the term "correct" signifies exactness. Additionally, based on J. Hadamard's article, the combination "correct task" denotes polite, tactful behavior. However, in order to grasp the meaning of the term "impolite", let's consider some examples.

Example 7. Let the following system be given (Equation 38):

$$\begin{cases} \xi+100\eta=111, \\ 100\xi+11011\eta=11111. \end{cases} \tag{38}$$

This equation has a unique solution, given by $\xi=1$ and $\eta=1$. Now, let's make a small transformation on the right side of the first Equation (37), and get the following (Equation 39):

$$\begin{cases} \xi+100\eta=111{,}11, \\ 100\xi+11011\eta=11111. \end{cases} \tag{39}$$

Similarly, this equation also has a unique solution, with $\xi=111{,}11$ and $\eta=0$. Therefore, a small change in the initial data of the problem leads to a significant change in its solution. We refer to such systems as "impolite" or ill-conditioned. These systems may describe real objects known to us with some error, and as a result, the solution to such a system may not exist.

Example 8. Now let's look at some simple examples. Let the following equations be given (Equation 40-43):

$$2^{\xi}=1024, \tag{40}$$

$$3^{\eta}=4, \tag{41}$$

$$2^{\varsigma}=-8, \tag{42}$$

$$1000^{\mu}=0. \tag{43}$$

Equation (39) has a unique solution equal to $\xi=10$. This means that the problem is correct.

Now let's pay attention to the Equation (40), which has no solutions to the set of rational numbers, i.e., is incorrect. If we expand the class of the considered solutions, i.e., add irrational numbers, then the problem becomes correct and its solution will be equal to $\eta=\log_3 4$.

Equation (41) is also incorrect, i.e., has no solutions for the set of real numbers. And if we mean a complex solution, then we get the following (Eqaution 44):

$$\varsigma=\log_2(-8)=\frac{\ln 8+i(1+2n)\pi}{\ln 8}, \ \ n\in Z. \tag{44}$$

Here, uniqueness is achieved if we mean single-valued branches of a multi-valued analytic function $w=\log_2\varsigma$. And finally, Equation (42) is also incorrect; here it is impossible to eliminate the incorrectness of the problem.

4. Conclusion

This paper shows how ill-posed problems arise, explains how estimation and inference can be carried out in ill-posed settings, and explains why estimation in these settings is important to study problems. The paper focusses examples that illustrate the issues and methods associated with ill-posed problems.

In conclusion, this work primarily focuses on the essence of incorrectly-posed problems in applied physical problems of mathematical physics. It acknowledges the historical appearance and development of incorrectly-posed problems in mathematical physics equations. Notable contributions from renowned scientists such as Carleman [1], Lavrent'ev [2-4], Tikhonov [5-7], Hadamard [8], Yarmukhamedov [9-12], Aizenberg [13], Tarkhanov [14-17], Arbuzov and Bukhgeim [18-21], and others have significantly advanced the field of solving incorrectly-posed problems. Such problems arise precisely in the processing of the results of physical experiments, which are directly carried out in such areas as astrophysics, geophysics and nuclear physics. Furthermore, they have valuable applications in fields like medicine, specifically in computed tomography. It is worth noting that despite the belief that ill-posed problems lacked physical meaning and were unsolvable, this notion has been disproven. Given the wide range of practical problems involving ill-posed problems, it has become imperative to address them and develop new methods for their solution.

The main contribution of this work is that, based on the results obtained from previous works for well-posed and ill-posed problems, it provides the necessary information to scientific researchers. For specificity, several simple examples are given, which give a clear idea of correct and incorrect problems. This means that solving ill-posed problems is more difficult than correct problems. That is, this means that we need to find approximate solutions. An ill posed problem will often need to be regularized or reformulated before you can give it a full numerical analysis using computer algorithms or other computational methods. Reformulation often involves bringing in new assumptions to fully define the problem and narrow it down. Tikhonov Regularization (sometimes called Tikhonov-Phillips regularization) is a popular way to deal with linear discrete ill-posed problems, which violate one of the terms of a well posed problem. Regularization stabilizes ill--posed problems, giving accurate approximate solutions-often by including prior information.

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Author contributions

Davron Aslonqulovich Juraev: Conceptualization, Methodology, Software. **Ebrahim Eldesoky Elsayed:** Data curation, Writing-Original draft preparation, Software, Validation. **Juan Jose Diaz Bulnes:** Visualization, Investigation, Writing-Reviewing and Editing. **Praveen Agarwal:** Visualization, Investigation, Writing-Reviewing and Editing. **Rostam Karim Saeed:** Writing-Reviewing and Editing.

Conflicts of interest

The authors declare no conflicts of interest.

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