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Approximate solutions of the Helmholtz equation on the plane

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Abstract

The study delves into the persistent analysis and reliability assessment of the solution to the Cauchy problem for the Helmholtz equation, within a defined area, using the known values on a smooth portion of the area's boundary as a reference. This analyzed situation belongs to those found within mathematical physics where the detected solutions fail to show consistent reliance on the initial data. Emphasizing real world applications, it's not only finding an approximate solution that matters, but also identifying its derivative. Presuming a solution is available and constantly differentiable within a close range, precise Cauchy data is considered. An explicit formula to expand both the solution and its derivative has been established, along with a regularization formula for instances where, under given conditions, ongoing approximations of initial Cauchy data with a defined error limit in the uniform metric are offered rather than the original data. Evaluations that assure the stability of the classical Cauchy problem solution have been provided.

1. Introduction

The research concentrates on the exploration and assessment of the stability and continuation of solutions for the Cauchy problem in relation to the Helmholtz equation. This is conducted within a domain, deriving data from existing values on a smooth part of the domain's boundary. This problem fits into broader issues of mathematical physics where solutions do not have a continuous dependence on the introductory data. In the undertaking of practical problems, it is not just about determining an approximate solution, but also its derivative. It is perceived that a solution to the problem exists and is continuously differentiable within a closed domain that precisely aligns with the Cauchy data. An explicit formula for the continuation of the solution and its derivative is established under these conditions. Moreover, a regularization formula is set out for instances when continuous approximations with a specific uniform metric error are given, as an alternative to the original Cauchy data. Certainty measures for the classical interpretation of the Cauchy problem solution have also been achieved. The problem under inspection falls into the category of ill-posed problems within mathematical physics. Tikhonov [1] clarified the intrinsic nature of such problems, highlighting their practical importance and noting that by restricting potential solution classes to a compact set, the solution's stability can be ascertained through existence and uniqueness, thereby rendering the task sustainable. Formulas empowering the discovery of the elliptic equation's solution when only partial Cauchy data is known are referred to as Carleman-type formulas. Carleman [2] established such a formula, presenting a solution to the Cauchy-Riemann equations in a domain of specific form. This idea was further developed by Goluzin and Krylov [3] who derived a formula that enables the determination of analytic functions' values using data known only on boundary segments. Pivotal to this research, and the body of similar works, is the Helmholtz equation, which has different solutions depending on the spaces it operates within. Renowned mathematician Yarmukhamedov [4] in 1977 proposed a method to construct a family of fundamental solutions. This method forms explicit formulas to recover solutions of elliptic problems from their Cauchy data on a section of the boundary. Formulas that have such properties as known are called formulas or Carleman matrices [4-7].

The Cauchy problem for most elliptic equations has a unique solution, i.e., this problem is solvable for an everywhere dense data set of the problem, but this data set is not considered closed. It follows that the theory of solvability of similar problems are very complex. Many problems of an applied nature, such as geo and biophysical, electrodynamics, gas, hydro and aerodynamic, problems of plasma physics, etc., are reduced to the equations of mathematical physics. At present, special attention is paid to topical aspects of differential equations and mathematical physics, which have scientific and practical applications in the fundamental sciences. In particular, special attention is paid to the study of various ill-posed boundary value problems for partial differential equations of elliptic type, which have practical application in applied sciences. As a result, significant results were obtained in studies of ill-posed boundary value problems for partial differential equations, that is, approximate solutions were constructed using Carleman matrices in explicit form from approximate data in special domains, estimates of conditional stability and solvability criteria were established.

A fundamental solution of the Helmholtz equations in a three-dimensional domain was constructed by Juraev [8]. Approximate solutions of the Cauchy problem for matrix factorization of the Helmholtz equation in two- and three-dimensional bounded and unbounded domains were constructed in papers [9-19]. Based on the results of previous works, regularization solutions for matrix factorizations of the Helmholtz operator were constructed for higher-order spaces [20-24]. Juraev, et al. [25-28], using the Carleman matrix, constructed an approximate solution to the Cauchy problem for various areas. Based on these results, a regularized solution of the Cauchy problem was found explicitly for various factorizations of the Helmholtz operator [29–34]. In works [35-36], new ways were found to solve some physical problems described by second-order ODEs of a special structure. Various boundary value problems for the Schrödinger equations, integro-differential equations, as well as for the Klein-Gordon equations are considered in papers [37-42]. You can also get acquainted with some boundary value problems in detail in [43-50]. Nonlocal boundary value problems for a loaded parabolic-hyperbolic equation were considered by Isomov and Kholbekov [51-52]. Boundary value problems for various modified equations, as well as for integral-differential equations, were studied in papers [53-60]. Dzhuraev [61] constructed an algorithm for constructing approximate solutions of the Cauchy problem for the Helmholtz equation.

The second section deals with the basic information that is required when posing the Cauchy problem. Based on the integral formula, the corresponding theorem for solving the Cauchy problem is proved in detail, and the results are generalized into two corollaries. In the third section of the work under study, it is proven that the resulting solution is stable. At the same time, two corresponding theorems are proven in detail. Based on Green's integral formula, approximate solutions of the Cauchy problem for the Helmholtz equations are obtained. Finally, the conclusion section summarizes the results obtained.

2. Basic Information and Problem Statement

This research is also preoccupied with the creation of the Helmholtz equation's fundamental solution family, parameterized by an entire function that possesses certain attributes.

Suppose R^2 be a real Euclidean space, $\xi=(\xi_1,\xi_2)\in R^2, \eta=(\eta_1,\eta_2)\in R^2, \alpha=|\eta_1-\xi_1|,\ r=|\eta-\xi|$.

Let's assume that $\Theta\subset R^2$ represents a real Euclidean space, and let $\Omega\subset R^2$ be a bounded simply connected domain with a piecewise smooth boundary composed of the plane $\Omega:\eta_2=0$ and a smooth curve Σ in the half-plane $\eta_2>0$, denoted as $\partial\Theta=\Sigma\bigcup\Omega$.

Supposing that $\Re(w)$ is an entire function taking real values for real w ($w=u+iv$, where u and v are real), and satisfying the condition in Equation 1:

$$\Re(u)\neq 0,\ \sup_{v\geq 1}\left|v^p\Re^{(p)}(w)\right|=K(u,p)<\infty,\\ u\in(-\infty,\infty),\ p=\overline{0,2}. \tag{1}$$

Let the function $\Psi(\eta,k;\xi)$ be defined by the following equality (Equation 2):

$$\Psi(\eta,\xi;\Lambda)=-\frac{1}{2\pi\Re(\xi_2)}\int_0^\infty \mathrm{Im}\frac{\Re(w)}{w-\xi_2}\frac{u\Im_0(\Lambda u)}{\sqrt{u^2+\alpha^2}}du,\\ \eta\neq\xi,\ w=i\sqrt{u^2+\alpha^2}+\eta_2, \tag{2}$$

here $\Im_0(\Lambda u)-$ is the Bessel function.

The function $\Psi(\eta,\xi;\Lambda)$ can be expressed as Equation 3.

$$\Psi(\eta,\xi;\Lambda)=\vartheta(\Lambda r)+\psi(\eta,\xi;\Lambda). \tag{3}$$

where $\vartheta(\Lambda r)=-\frac{i}{4}H_0^{(1)}(\Lambda r)$ is the fundamental solution, $\psi(\eta,\xi;\Lambda)-$ is the regular solution.

We consider the Helmholtz equation in the domain Θ given by Equation 4.

$$\Delta W(\eta)+\Lambda^2 W(\eta)=0, \tag{4}$$

here Λ is the positive number, $\Delta=\sum_{j=1}^{2}\partial_{\eta_j}^2$.

2.1. The Cauchy problem. Suppose $W(\eta)\in C^2(\Theta)\bigcap C^1(\Theta)$ and (Equation 5)

$$W(\eta)\big|_S = f(\eta),\ \frac{\partial W(\eta)}{\partial n}\bigg|_S = g(\eta),\quad \eta\in S. \tag{5}$$

The Cauchy problem involves specific initial data and aims to determine the function $W(\eta)=W(\eta_1,\eta_2)\in C^2(\Theta)\bigcap C^1(\Theta)$ in Θ based on its values $f(\eta)$ и $g(\eta)$ on the boundary $\partial\Theta$. The main objective is to find an approximate solution to the Helmholtz equation while assessing the stability of the solutions within the context of the Cauchy problem. As the deductions are further refined, it will provide practical solutions, leading to tangible benefits for mathematical physics and various natural science fields.

In Equation 2, choosing the replacement (Equation 6)

$$\Re(w)=e^{\lambda w},\ \Re(\xi_2)=e^{\lambda\xi_2},\ \lambda>0, \tag{6}$$

we obtain the following integral representation (Equation 7)

$$\Psi_\lambda(\eta,\xi;\Lambda)=-\frac{e^{-\lambda\xi_2}}{2\pi}\int_0^\infty \operatorname{Im}\frac{e^{\lambda w}}{w-\xi_2}\frac{u\Im_0(\Lambda u)}{\sqrt{u^2+\alpha^2}}du,$$
$$\lambda\geq\Lambda+\lambda_0,\ \lambda_0>0. \tag{7}$$

For a function $W(\eta)\in C^2(\Theta)\bigcap C^1(\Theta)$ and any $\xi\in\Theta$, Green's integral formula (Equation 8) holds:

$$W(\xi)=\int_{\partial\Theta}\left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right]ds_\eta,\ \xi\in\Theta, \tag{8}$$

Theorem 1. Let $W(\eta)\in C^2(\Theta)\bigcap C^1(\Theta)$ and (Equation 9)

$$|W(\eta)|+|\partial_n W(\eta)|\leq K,\ \eta\in\Omega. \tag{9}$$

If (Equation 10)

$$W_\lambda(\xi)=\int_\Sigma\left[g(\eta)\Psi_\lambda(\eta,\xi;\Lambda)-f(\eta)\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right]ds_\eta,\ \xi\in\Theta, \tag{10}$$

then the following estimates (Equation 11-12) are true

$$|W(\xi)-W_\lambda(\xi)|\leq C(\Lambda,\xi)\lambda Ke^{-\lambda\xi_2},\ \lambda>1,\ \xi\in\Theta, \tag{11}$$

$$\left|\partial_{\xi_j}W(\xi)-\partial_{\xi_j}W_\lambda(\xi)\right|\leq C(\Lambda,\xi)\lambda Ke^{-\lambda\xi_2},\ \lambda>1,\ \xi\in\Theta,\ j=\overline{1,2}. \tag{12}$$

where $C(\Lambda,\xi)$ and further, denotes bounded functions on compact subsets of Θ.

Proof. Let's first establish the estimate (Equation 11). By applying the integral formula (Equation 8) and the Equation 10, we can derive Equation 13.

$$\begin{aligned} W(\xi) &= \int_{\partial\Theta} \left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda) - W(\eta)\partial_n \Psi_\lambda(\eta,\xi;\Lambda)\right] ds_\eta = \\ &= \int_{\Sigma} \left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda) - W(\eta)\partial_n \Psi_\lambda(\eta,\xi;\Lambda)\right] ds_\eta + \\ &+ \int_{\Omega} \left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda) - W(\eta)\partial_n \Psi_\lambda(\eta,\xi;\Lambda)\right] ds_\eta = \\ &= W_\lambda(\xi) + \int_{\Omega} \left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda) - W(\eta)\partial_n \Psi_\lambda(\eta,\xi;\Lambda)\right] ds_\eta, \quad \xi \in \Theta. \end{aligned} \tag{13}$$

Considering the inequality (Equation 9), we can estimate Equation 14.

$$\begin{aligned} &\left|W(\xi) - W_\lambda(\xi)\right| \le \\ &\le \left|\int_{\Omega} \left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda) - W(\eta)\partial_n \Psi_\lambda(\eta,\xi;\Lambda)\right] ds_\eta\right| \le \\ &\le K\int_{\Omega} \left[\left|\Psi_\lambda(\eta,\xi;\Lambda)\right| + \left|\partial_n \Psi_\lambda(\eta,\xi;\Lambda)\right|\right] ds_\eta, \quad \xi \in \Theta. \end{aligned} \tag{14}$$

For the proof, we will evaluate the integrals $\int_{\Omega} \left|\Psi_\lambda(\eta,\xi;\Lambda)\right| ds_\eta$, $\int_{\Omega} \left|\partial_{\eta_1} \Psi_\lambda(\eta,\xi;\Lambda)\right| ds_\eta$ and $\int_{\Omega} \left|\partial_{\eta_2} \Psi_\lambda(\eta,\xi;\Lambda)\right| ds_\eta$ on the part Ω, i.e., on $\eta_2 = 0$.

To do this, we will select the imaginary part of identity (Equation 7), we obtain the Equation 15.

$$\begin{aligned} \Psi_\lambda(\eta,\xi;\Lambda) &= \frac{e^{\sigma(\eta_2-\xi_2)}}{2\pi}\left[\int_0^\infty \frac{\cos\lambda\sqrt{u^2+\alpha^2}}{u^2+r^2} u\Im_0(\Lambda u)\,du - \right. \\ &\left. - \int_0^\infty \frac{\eta_2 \sin\lambda\sqrt{u^2+\alpha^2}}{u^2+r^2} \frac{u\Im_0(\Lambda u)}{\sqrt{u^2+\alpha^2}}\,du, \quad \eta \ne \xi,\ \xi_2 > 0. \right. \end{aligned} \tag{15}$$

Based on (Equation 14), as well as the inequality (Equation 16)

$$\Im_0(\Lambda u) \le \sqrt{\frac{2}{\Lambda\pi u}}, \tag{16}$$

we have the following (Equation 17)

$$\int_{\Omega} \left|\Psi_\lambda(\eta,\xi;\Lambda)\right| ds_\eta \le C(\Lambda,\xi)\lambda e^{-\lambda\xi_2}, \quad \lambda > 1,\ \xi \in \Theta. \tag{17}$$

Now, we estimate the second integral given the following (Equation 18)

$$\partial_{\eta_1}\Psi_\lambda(\eta,\xi;\Lambda) = \partial_s \Psi_\lambda(\eta,\xi;\Lambda)\partial_{\eta_1} s = 2(\eta_1 - \xi_1)\partial_s \Psi_\lambda(\eta,\xi;\Lambda). \tag{18}$$

Based on equality (Equation 14), inequality (Equation 15), and taking into account equality (Equation 17), we have (Equation 19).

$$\int_{\Omega} \left|\partial_{\eta_1}\Psi_\lambda(\eta,\xi;\Lambda)\right| ds_\eta \le C(\Lambda,\xi)\lambda e^{-\lambda\xi_2}, \quad \lambda > 1,\ \xi \in \Theta. \tag{19}$$

Now, we estimate the integral $\int_{\Omega} \left|\partial_{\eta_2}\Psi_\lambda(\eta,\xi;\Lambda)\right| ds_\eta$.

Considering equality (Equation 14) and inequality (Equation 15), we have Equation 20.

$$\int_{\Omega}\left|\partial_{\eta_2}\Psi_{\lambda}(\eta,\Lambda;\xi)\right|ds_{\eta}\leq C(\Lambda,\xi)\lambda e^{-\lambda\xi_2},\ \ \lambda>1,\ \xi\in\Theta. \tag{20}$$

Finally, by combining the estimates of (Equation 12,13,16) and incorporating (Equation 1), we obtain the estimate (Equation 11).

To prove inequality (Equation 12), we differentiate Equations (8) and (10). In this case, differentiation is taken according to ξ_j, $j=\overline{1,2}$ (Equation 21).

$$\begin{gathered}\partial_{\xi_j}W(\xi)=\int_{\partial\Theta}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right]ds_{\eta}=\\ =\int_{\Sigma}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right]ds_{\eta}+\\ +\int_{\Omega}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right]ds_{\eta},\\ \partial_{\xi_j}W_{\lambda}(\xi)=\int_{S}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right]ds_{\eta},\\ \xi\in\Theta,\ \ j=\overline{1,2}.\end{gathered} \tag{21}$$

From the upper equality, as well as applying inequality (Equation 9), we can estimate (Equation 22):

$$\begin{gathered}\left|\partial_{\xi_j}W(\xi)-\partial_{\xi_j}W_{\lambda}(\xi)\right|\leq\\ \leq\int_{\Omega}\left[\left|\partial_n W(\eta)\partial_{\xi_j}\right|\left|\Psi_{\lambda}(\eta,\xi;\Lambda)\right|-\left|W(\eta)\right|\left|\partial_{\xi_j}\left(\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right|\right]ds_{\eta}\leq\\ \leq K\int_{\Omega}\left[\left|\partial_{\xi_j}\Psi_{\lambda}(\eta,\xi;\Lambda)\right|+\left|\partial_{\xi_j}\left(\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right|\right]ds_{\eta},\ \ \xi\in\Theta,\ \ j=\overline{1,2}.\end{gathered} \tag{22}$$

For the proof, we will evaluate the integrals $\int_{\Omega}\left|\partial_{\xi_1}\Psi_{\lambda}(\eta,\xi;\Lambda)\right|ds_{\eta}$ and $\int_{\Omega}\left|\partial_{\xi_2}\Psi_{\lambda}(\eta,\xi;\Lambda)\right|ds_{\eta}$ on the part Ω of the plane $\eta_2=0$.

Here, to estimate the first integral, taking into account (Equation 23)

$$\partial_{\xi_1}\Psi_{\lambda}(\eta,\xi;\Lambda)=\partial_s\Psi_{\lambda}(\eta,\xi;\Lambda)\partial_{\xi_1}s=-2(\eta_1-\xi_1)\partial_s\Psi_{\lambda}(\eta,\xi;\Lambda), \tag{23}$$

as well as equality (Equation 14), inequality (Equation 15), and equality (Equation 22), we obtain the estimate (Equation 24).

$$\int_{\Omega}\left|\partial_{\xi_1}\Psi_{\lambda}(\eta,\Lambda;\xi)\right|ds_{\eta}\leq C(\Lambda,\xi)\lambda e^{-\lambda\xi_2},\ \ \lambda>1,\ \xi\in\Theta. \tag{24}$$

Now, we estimate the integral $\int_{\Omega}\left|\partial_{\xi_2}\Psi_{\lambda}(\eta,\xi;\Lambda)\right|ds_{\eta}$ (Equation 25).

$$\int_{\Omega}\left|\partial_{\xi_2}\Psi_{\lambda}(\eta,\xi;\Lambda)\right|ds_{\eta}\leq C(\Lambda,\xi)\lambda e^{-\lambda\xi_2},\ \ \lambda>1,\ \xi\in\Theta. \tag{25}$$

Finally, by combining the estimates of (Equation 21), (Equation 12), and taking into account (Equation 21), we obtain the inequality (Equation 12).

Corollary 1. For any $\xi\in\Theta$, the following limit relations are true (Equation 26):

$$\lim_{\lambda\to\infty}W_{\lambda}(\xi)=W(\xi),\ \ \lim_{\lambda\to\infty}\partial_{\xi_j}W_{\lambda}(\xi)=\partial_{\xi_j}W(\xi),\ \ \xi\in\Theta,\ \ j=\overline{1,2}. \tag{26}$$

Assume that $\overline{\Theta}_\varepsilon$ consists of a set We denote by $\overline{\Theta}_\varepsilon$ the set (Equation 27):

$$\overline{\Theta}_\varepsilon = \{(\xi_1,\xi_2)\in\Theta,\ a>\xi_2\ge\varepsilon,\ a=\max_{\Omega}\psi(\xi_1),\ 0<\varepsilon<a\}. \tag{27}$$

In this case, $\overline{\Theta}_\varepsilon \subset \Theta$ is considered a compact set.

Corollary 2. Suppose $\xi\in\overline{\Theta}_\varepsilon$, in this case the families of functions $\{W_\sigma(\xi)\}$, $\left\{\partial_{\xi_j}W_\sigma(\xi)\right\}$ converge uniformly as $\sigma\to\infty$ (Equation 28).

$$W_\lambda(\xi)\ \text{Ã}\ W(\xi),\ \ \partial_{\xi_j}W_\lambda(\xi)\ \text{Ã}\ \partial_{\xi_j}W(\xi),\ \ j=\overline{1,2}. \tag{28}$$

In this case, the set $E_\varepsilon=\Theta\setminus\overline{\Theta}_\varepsilon$ is a boundary layer for this problem, similar to the theory of singular perturbations, in which uniform convergence cannot exist.

2.2. Sustainability Assessment

Let the following equation be given on a smooth curve Σ, i.e., (Equation 29).

$$\eta_2=\psi(\eta_1),\ \ \eta_1\in(-\infty,\infty). \tag{29}$$

Let's put (Equation 30)

$$a=\max_{\Omega}\psi(\eta_1),\ \ b=\max_{\Omega}\sqrt{1+\left(\frac{d\psi}{d\eta_1}\right)^2} \tag{30}$$

Theorem 2. Suppose $W(\eta)\in C^2(\Theta)\bigcap C^1(\Theta)$ satisfy boundary condition (9), and and also on Σ the following inequality (Equation 31)

$$|W(\eta)|+|\partial_n W(\eta)|\le\mu,\ \eta\in\Sigma. \tag{31}$$

At the same time, the following are true (Equation 32 and 33)

$$|W(\xi)|\le C(\Lambda,\xi)\lambda K^{1-\frac{\xi_2}{a}}\mu^{\frac{\xi_2}{a}},\ \ \lambda>1,\ \ \xi\in\Theta, \tag{32}$$

$$\left|\partial_{\xi_j}W(\xi)\right|\le C(\Lambda,\xi)\lambda K^{1-\frac{\xi_2}{a}}\mu^{\frac{\xi_2}{a}},\ \ \lambda>1,\ \ \xi\in\Theta,\ \ j=\overline{1,2}. \tag{33}$$

Proof. Initially, we will estimate inequality (Equation 26). Based on the integral representation (Equation 8), we obtain the Equation 34.

$$\begin{aligned} W(\xi) &= \int_{\Sigma}\left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right]ds_\eta + \\ &+\int_{\Omega}\left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right]ds_\eta,\ \ \xi\in\Theta. \end{aligned} \tag{34}$$

We will evaluate the equality (Equation 34) and obtain Equation 35.

$$\begin{aligned} |W(\xi)| &\le \left|\int_{\Sigma}\left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right]ds_\eta\right| + \\ &+\left|\int_{\Omega}\left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right]ds_\eta\right|,\ \ \xi\in\Theta. \end{aligned} \tag{35}$$

Based on inequality (Equation 26), we first estimate the integral, where the integration is considered on Σ, i.e., (Equation 36).

$$\begin{gathered}\left|\int_{\Sigma}\left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda) - W(\eta)\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right]ds_\eta\right| \le \\ \le \int_{\Sigma}\left[\left|\partial_n W(\eta)\right|\left|\Psi_\lambda(\eta,\xi;\Lambda)\right| - \left|W(\eta)\right|\left|\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right|\right]ds_\eta \le \\ \le \mu\int_{\Sigma}\left[\left|\Psi_\lambda(\eta,\xi;\Lambda)\right| + \left|\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right|\right]ds_\eta,\ \xi\in\Theta.\end{gathered} \tag{36}$$

For the proof, we will evaluate the integrals $\int_{\Sigma}\left|\Psi_\lambda(\eta,\xi;\Lambda)\right|ds_\eta$, $\int_{\Sigma}\left|\partial_{\eta_1}\Psi_\lambda(\eta,\xi;\Lambda)\right|ds_\eta$ and $\int_{\Sigma}\left|\partial_{\eta_2}\Psi_\lambda(\eta,\xi;\Lambda)\right|ds_\eta$ on a smooth curve, i.e., on Σ.

Based on equality (Equation 14) and inequality (Equation 15), we have the Equation 37.

$$\int_{\Sigma}\left|\Psi_\lambda(\eta,\xi;\Lambda)\right|ds_\eta \le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta. \tag{37}$$

Based on equalities (Equation 14) and (Equation 17), as well as inequality (Equation 15), estimating the second integral, we will have the corresponding estimate (Equation 38)

$$\int_{\Sigma}\left|\partial_{\eta_1}\Psi_\lambda(\eta,\xi;\Lambda)\right|ds_\eta \le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta. \tag{38}$$

To estimate the integral $\int_{\Sigma}\left|\partial_{\eta_2}\Psi_\lambda(\eta,\xi;\Lambda)\right|ds_\eta$, from (Equation 14) – (Equation 15), respectively, we have Equation 39.

$$\int_{\Sigma}\left|\partial_{\eta_2}\Psi_\lambda(\eta,\xi;\Lambda)\right|ds_\eta \le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta. \tag{39}$$

From (Equation 31) - (Equation 33), bearing in mind (Equation 30), we obtain Equation 40.

$$\begin{gathered}\left|\int_{\Sigma}\left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda) - W(\eta)\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right]ds_\eta\right| \le \\ \le \mu\int_{\Sigma}\left[\left|\Psi_\lambda(\eta,\xi;\Lambda)\right| + \left|\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right|\right]ds_\eta \le \\ \le C(\Lambda,\xi)\lambda\mu e^{\lambda(a-\xi_2)},\ \lambda>1,\ \xi\in\Theta.\end{gathered} \tag{40}$$

The following is known (Equation 41).

$$\begin{gathered}\left|W(\xi) - W_\lambda(\xi)\right| \le \\ \le \left|\int_{\Omega}\left[\partial_n W(\eta)\Psi_\lambda(\eta,\xi;\Lambda) - W(\eta)\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right]ds_\eta\right| \le \\ \le \int_{\Omega}\left[\left|\Psi_\lambda(\eta,\xi;\Lambda)\right| + \left|\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right|\right]ds_\eta \le \\ \le C(\Lambda,\xi)\lambda K e^{-\sigma\xi_2},\ \lambda>1,\ \xi\in\Theta.\end{gathered} \tag{41}$$

Finally, using the obtained estimates (Equation 34) - (Equation 35), and also taking into account (Equation 29), we obtain the Equation 42.

$$\left|W(\xi)\right| \le \frac{C(\Lambda,\xi)\lambda}{2}\left(\mu e^{\lambda a} + K\right)e^{-\lambda\xi_2},\ \lambda>1,\ \xi\in\Theta. \tag{42}$$

Choosing λ from the Equation 43:

$$\lambda = \frac{1}{a}\ln\frac{K}{\mu}, \tag{43}$$

completely we get the estimate (Equation 26).

Now we turn to inequality (Equation 27). It is required to take partial derivatives of the integral representation (Equation 8). Here the integration is carried out according to ξ_j, $j=\overline{1,2}$ (Equation 44).

$$\begin{gathered}\partial_{\xi_j}W(\xi)=\int\limits_{\partial\Theta}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right]ds_\eta=\\ =\int\limits_{\Sigma}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right]ds_\eta+\\ +\int\limits_{\Omega}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right]ds_\eta=\\ =\partial_{\xi_j}W_\lambda(\xi)+\int\limits_{\Omega}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right]ds_\eta,\\ \xi\in\Theta,\quad j=\overline{1,2}.\end{gathered} \tag{44}$$

Here we obtain (Equation 45)

$$\begin{gathered}\partial_{\xi_j}W_\lambda(\xi)=\int\limits_{\Sigma}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right]ds_\eta,\\ \xi\in\Theta,\quad j=\overline{1,2}.\end{gathered} \tag{45}$$

Based on identity (Equation 20) and boundary condition (Equation 9), we obtain the Equation 46.

$$\begin{gathered}\left|\partial_{\xi_j}W(\xi)\right|\le\left|\int\limits_{\partial\Theta}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right]ds_\eta\right|\le\\ \le\left|\int\limits_{\Sigma}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right]ds_\eta\right|+\\ +\left|\int\limits_{\Omega}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right]ds_\eta\right|\le\\ \le\left|\partial_{\xi_j}W_\lambda(\xi)\right|+\left|\int\limits_{\Omega}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right]ds_\eta\right|,\\ \xi\in\Theta,\quad j=\overline{1,2}.\end{gathered} \tag{46}$$

Taking into account condition (Equation 25), we estimate $\left|\partial_{\xi_j}W(\xi)\right|$ (Equation 47).

$$\begin{gathered}\left|\partial_{\xi_j}W(\xi)\right|\le\int\limits_{\Sigma}\left[\left|\partial_n W(\eta)\right|\left|\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)\right|-\left|W(\eta)\right|\left|\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right|\right]ds_\eta\le\\ \le\mu\int\limits_{\Sigma}\left[\left|\partial_{\xi_j}\Psi_\lambda(\eta,\xi;\Lambda)\right|+\left|\partial_{\xi_j}\left(\partial_n\Psi_\lambda(\eta,\xi;\Lambda)\right)\right|\right]ds_\eta,\ \ \xi\in\Theta,\quad j=\overline{1,2}.\end{gathered} \tag{47}$$

To do this, we estimate the integrals $\int\limits_{\Sigma}\left|\partial_{\xi_1}\Psi_\lambda(\eta,\xi;\Lambda)\right|ds_\eta$ and $\int\limits_{\Sigma}\left|\partial_{\xi_2}\Psi_\lambda(\eta,\xi;\Lambda)\right|ds_\eta$ on a smooth curve S.

Based on equality (Equation 14), inequality (Equation 17), and formula (Equation 22), we have (Equation 48).

$$\int\limits_{\Sigma}\left|\partial_{\xi_1}\Psi_\lambda(\eta,\xi;\Lambda)\right|ds_\eta\le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \ \lambda>1,\ \ \xi\in\Theta. \tag{48}$$

Finally, we estimate the integral $\int_{\Sigma}\left|\partial_{\xi_2}\Psi_{\lambda}(\eta,\xi;\Lambda)\right|ds_{\eta}$ In this case, on the basis of equality (Equation 14) and inequality (Equation 15), we obtain the Equation 49.

$$\int_{\Sigma}\left|\partial_{\xi_2}\Psi_{\lambda}(\eta,\xi;\Lambda)\right|ds_{\eta} \le C(\Lambda,\xi)\lambda e^{\lambda(a-\xi_2)},\ \ \lambda>1,\ \ \xi\in\Theta. \tag{49}$$

From (Equation 42) – (Equation 43), as well as (Equation 41), we have (Equation 50).

$$\left|\int_{\Sigma}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right]ds_{\eta}\right| \le C(\Lambda,\xi)\lambda\mu e^{\lambda(a-\xi_2)},$$
$$\lambda>1,\ \ \xi\in\Theta,\ \ j=\overline{1,2}. \tag{50}$$

It is known that we obtained the following estimate in the proof of the previous theorem (Equation 51).

$$\left|\int_{\Omega}\left[\partial_n W(\eta)\partial_{\xi_j}\Psi_{\lambda}(\eta,\xi;\Lambda)-W(\eta)\partial_{\xi_j}\left(\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right]ds_{\eta}\right| \le C(\Lambda,\xi)\lambda K e^{-\lambda\xi_2},\ \ \xi\in\Theta. \tag{51}$$

From (Equation 44) - (Equation 45), as well as (Equation 40), we will have Equation 52.

$$\left|\partial_{\xi_j}W(\xi)\right| \le \frac{C(\Lambda,\xi)\lambda}{2}(\mu e^{\lambda a}+K)e^{-\lambda\xi_2},\ \ \lambda>1,\ \ \xi\in\Theta,\ \ j=\overline{1,2}. \tag{52}$$

In estimate (Equation 27), choosing the parameter σ from equality (Equation 37), we finally prove completeness (Equation 27).

Suppose $W(\eta)\in C^2(\Theta)\bigcap C^1(\Theta)$ and instead of functions $f(\eta),\ g(\eta)$ on a smooth curve Σ be given by their approximations $f_{\mu}(\eta),\ g_{\mu}(\eta)$, respectively, with an error of $0<\mu<1$ (Equation 53).

$$\max_{\Sigma}\left|f(\eta)-f_{\mu}(\eta)\right|\le\mu,\ \ \max_{\Sigma}\left|g(\eta)-g_{\mu}(\eta)\right|\le\mu. \tag{53}$$

We put (Equation 54):

$$W_{\lambda(\mu)}(\xi)=\int_{\Sigma}\left[g_{\mu}(\eta)\Psi_{\lambda}(\eta,\xi;\Lambda)-f_{\mu}(\eta)\partial_n\Psi_{\mu}(\eta,\xi;\Lambda)\right]ds_{\eta},\ \ \xi\in\Theta. \tag{54}$$

Theorem 3. Let $W(\eta)\in C^2(\Theta)\bigcap C^1(\Theta)$ on the part of the plane $\Omega:\eta_2=0$ satisfying condition (Equation 9), then the Equations 55 and 56:

$$\left|W(\xi)-W_{\lambda}(\xi)\right|\le C(\Lambda,\xi)\lambda K^{1-\frac{\xi_2}{a}}\mu^{\frac{\xi_2}{a}},\ \ \lambda>1,\ \ \xi\in\Theta, \tag{55}$$

$$\left|\partial_{\xi_j}W(\xi)-\partial_{\xi_j}W_{\lambda}(\xi)\right|\le C(\Lambda,\xi)\lambda K^{1-\frac{\xi_2}{a}}\mu^{\frac{\xi_2}{a}},\ \ \lambda>1,\ \ \xi\in\Theta,\ \ j=\overline{1,2}. \tag{56}$$

Proof. Taking into account the integral representations (Equation 8) and (Equation 48), we have Equation 57.

$$\begin{aligned} W(\xi)-W_{\lambda(\mu)}(\xi)== &\int_{\Sigma}\left[g(\eta)\Psi_{\lambda}(\eta,\xi;\Lambda)-f(\eta)\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right]ds_{\eta}+\\ &+\int_{\Omega}\left[g(\eta)\Psi_{\lambda}(\eta,\xi;\Lambda)-f(\eta)\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right]ds_{\eta}-\\ &-\int_{\Sigma}\left[g_{\mu}(\eta)\Psi_{\lambda}(\eta,\xi;\Lambda)-f_{\mu}(\eta)\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right]ds_{\eta}=\\ =-\int_{\Sigma}\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\{f(\eta)-f_{\mu}(\eta)\}ds_{\eta}&+\int_{\Sigma}\Psi_{\sigma}(\eta,\xi;\Lambda)\{g(\eta)-g_{\mu}(\eta)\}ds_{\eta}+\\ &+\int_{\Omega}\left[g(\eta)\Psi_{\lambda}(\eta,\xi;\Lambda)-f(\eta)\partial_n\Psi_{\lambda}(\eta,\xi;\Lambda)\right]ds_{\eta},\ \ \xi\in\Theta. \end{aligned} \tag{57}$$

And obtain Equation 58.

$$\begin{aligned}
\partial_{\xi_j} W(\xi) - \partial_{\xi_j} W_{\lambda(\mu)}(\xi) &= \int_{\Sigma} \left[g(\eta)\partial_{\xi_j} \Psi_{\lambda}(\eta,\xi;\Lambda) - f(\eta)\partial_{\xi_j} \left(\partial_n \Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right] ds_{\eta} + \\
&+\int_{\Omega} \left[g(\eta)\partial_{\xi_j} \Psi_{\lambda}(\eta,\xi;\Lambda) - f(\eta)\partial_{\xi_j} \left(\partial_n \Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right] ds_{\eta} - \\
&-\int_{\Sigma} \left[g_{\mu}(\eta)\partial_{\xi_j} \Psi_{\sigma}(\eta,\xi;\Lambda) - f_{\mu}(\eta)\partial_{\xi_j} \left(\partial_n \Psi_{\sigma}(\eta,\xi;\Lambda)\right)\right] ds_{\eta} = \\
&= -\int_{\Sigma} \partial_{\xi_j} \left(\partial_n \Psi_{\sigma}(\eta,\xi;\Lambda)\right)\{ f(\eta) - f_{\mu}(\eta)\} ds_{\eta} + \int_{\Sigma} \partial_{\xi_j} \Psi_{\lambda}(\eta,\xi;\Lambda)\{g(\eta) - g_{\delta}(\eta)\} ds_{\eta} + \\
&+\int_{\Omega} \left[g(\eta)\partial_{\xi_j} \Psi_{\lambda}(\eta,\xi;\Lambda) - f(\eta)\partial_{\xi_j} \left(\partial_n \Psi_{\lambda}(\eta,\xi;\Lambda)\right)\right] ds_{\eta}, \quad \xi \in \Theta, \quad j = \overline{1,2}.
\end{aligned} \tag{58}$$

Further, from the boundary condition (Equation 9) and conditions (Equation 47), we will, respectively, evaluate the Equation 59.

$$\begin{aligned}
\left| W(\xi) - W_{\lambda(\mu)}(\xi) \right| &\le \int_{\Sigma} \left| \partial_n \Psi_{\lambda}(\eta,\xi;\Lambda) \right| \left| \{ f(\eta) - f_{\mu}(\eta)\} \right| ds_{\eta} + \\
&+\int_{\Sigma} \left| \Psi_{\sigma}(\eta,\xi;\Lambda) \right| \left| \{ g(\eta) - g_{\mu}(\eta)\} \right| ds_{\eta} + \\
&+\int_{\Omega} \left[\left| g(\eta) \right| \left| \Psi_{\lambda}(\eta,\xi;\Lambda) \right| - \left| f(\eta) \right| \left| \partial_n \Psi_{\lambda}(\eta,\xi;\Lambda) \right| \right] ds_{\eta} \le \\
&\le \mu \int_{\Sigma} \left| \partial_n \Psi_{\lambda}(\eta,\xi;\Lambda) \right| ds_{\eta} + \mu \int_{\Sigma} \left| \Psi_{\sigma}(\eta,\xi;\Lambda) \right| ds_{\eta} + \\
&+ K \int_{\Omega} \left[\left| \Psi_{\lambda}(\eta,\xi;\Lambda) \right| - \left| \partial_n \Psi_{\lambda}(\eta,\xi;\Lambda) \right| \right] ds_{\eta}, \quad \xi \in \Theta.
\end{aligned} \tag{59}$$

And obtain Equation 60.

$$\begin{aligned}
\left| \partial_{\xi_j} W(\xi) - \partial_{\xi_j} W_{\lambda(\mu)}(\xi) \right| &\le \int_{\Sigma} \left| \partial_{\xi_j} \left(\partial_n \Psi_{\sigma}(\eta,\xi;\Lambda)\right) \right| \left| \{ f(\eta) - f_{\mu}(\eta)\} \right| ds_{\eta} + \\
&+\int_{\Sigma} \left| \partial_{\xi_j} \Psi_{\lambda}(\eta,\xi;\Lambda) \right| \left| \{ g(\eta) - g_{\delta}(\eta)\} \right| ds_{\eta} + \\
&+\int_{\Omega} \left[\left| g(\eta) \right| \left| \partial_{\xi_j} \Psi_{\lambda}(\eta,\xi;\Lambda) \right| - \left| f(\eta) \right| \left| \partial_{\xi_j} \left(\partial_n \Psi_{\lambda}(\eta,\xi;\Lambda)\right) \right| \right] ds_{\eta} \le \\
&\le \mu \int_{\Sigma} \left| \partial_{\xi_j} \left(\partial_n \Psi_{\sigma}(\eta,\xi;\Lambda)\right) \right| ds_{\eta} + \mu \int_{\Sigma} \left| \partial_{\xi_j} \Psi_{\lambda}(\eta,\xi;\Lambda) \right| ds_{\eta} + \\
&+ K \int_{\Omega} \left[\left| \partial_{\xi_j} \Psi_{\lambda}(\eta,\xi;\Lambda) \right| - \left| \partial_{\xi_j} \left(\partial_n \Psi_{\lambda}(\eta,\xi;\Lambda)\right) \right| \right] ds_{\eta}, \quad \xi \in \Theta, \quad j = \overline{1,2}.
\end{aligned} \tag{60}$$

From the results of the theorems obtained above, we obtain, respectively, the Equation 61 and 62.

$$\left| W(\xi) - W_{\lambda(\mu)}(\xi) \right| \le \frac{C(\Lambda,\xi)\sigma}{2} (\mu e^{\lambda a} + K) e^{-\sigma \xi_2}, \quad \Lambda > 1, \quad \xi \in \Theta, \tag{61}$$

$$\left| \partial_{\xi_j} W(\xi) - \partial_{\xi_j} W_{\lambda(\mu)}(\xi) \right| \le \frac{C(\Lambda,\xi)\sigma}{2} (\mu e^{\lambda a} + K) e^{-\sigma \xi_2}, \quad \Lambda > 1, \quad \xi \in \Theta, \quad j = \overline{1,2}. \tag{62}$$

In the last obtained estimates (Equation 61) and (Eqaution 62), choosing the parameter λ, from (Equation 37) respectively, we obtain the proof of Theorem 3.

Corollary 3. We claim that for any $\xi \in \Theta$, the following limit Equation 63.

$$\lim_{\mu \to 0} W_{\lambda(\mu)}(\xi) = W(\xi), \quad \lim_{\mu \to 0} \partial_{\xi_j} W_{\lambda(\mu)}(\xi) = \partial_{\xi_j} W(\xi), \quad \xi \in \Theta, \quad j = \overline{1,2}. \tag{63}$$

Corollary 4. It turns out that if $\xi \in \overline{\Theta}_\varepsilon$, then the families of functions $\{W_{\lambda(\mu)}(\xi)\}$ and $\{\partial_{\xi_j} W_{\lambda(\mu)}(\xi)\}$ converge uniformly at $\mu \to 0$ (Equation 64).

$$W_{\lambda(\mu)}(\xi)\ \tilde{A}\ W(\xi),\ \partial_{\xi_j} W_{\lambda(\mu)}(\xi)\ \tilde{A}\ \partial_{\xi_j} W(\xi),\ \xi \in \Theta,\ j = \overline{1,2}. \tag{64}$$

3. Conclusion

In conclusion, this study employs the Carleman function to recover an unknown function based on Cauchy data provided on a specific portion of the boundary within the region. By constructing the Carleman function and applying Green's formula, an explicit regularized solution can be obtained. The research demonstrates that constructing the Carleman function efficiently is equivalent to constructing a regularized solution for the Cauchy problem. It is assumed that a differentiable solution to the problem exists within a closed domain with precisely defined Cauchy data. In this case, explicit formulas for the continuation of the solution and its derivative are established, along with a regularization formula for situations where continuous approximations of the initial Cauchy data are supplied with a specified error in the uniform metric. Furthermore, stability estimates for the solution of the Cauchy problem in the classical sense are also derived. In this work, based on the Carleman function, a regularized solution to the Cauchy problem for the Helmholtz equation in a two-dimensional bounded domain was constructed. The regularized solution defines a stable approximate solution method. Theorems on the stability of the solution are proved and the corresponding consequences of these theorems are obtained.

Thus, functionals $W_{\lambda(\mu)}(\xi)$ and $\partial_{\xi_j} W_{\lambda(\mu)}(\xi)$ determines the regularization of the solution of problem (Equation 4) - (Equation 5).

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Author contributions

Davron Aslonqulovich Juraev: Conceptualization, Methodology, Software. **Ali Shokri:** Data curation, Writing-Original draft preparation. **Praveen Agarwal:** Visualization, Investigation, Writing-Reviewing and Editing. **Praveen Agarwal:** Visualization, Investigation, Writing-Reviewing and Editing. **Ebrahim Eldesoky Elsayed:** Data curation, Software, Validation. **Irfan Nurhidayat:** Data curation, Writing-Original draft preparation, Software, Validation.

Conflicts of interest

The authors declare no conflicts of interest.

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