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On the formulation of the Cauchy problem for matrix factorizations of the Helmholtz equation

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Abstract
In this paper, we are talking about the formulation of the Cauchy problem for matrix factorizations of the Helmholtz equation in two-dimensional and three-dimensional bounded domains. Preliminary information and formulation of the Cauchy problem are given. The corresponding examples characterizing the matrix factorization of the Helmholtz equation are constructed. On the basis of the constructed Carleman function, a regularized solution of the Cauchy problem for the matrix factorization of the Helmholtz equation on the plane in three-dimensional bounded domains is constructed in an explicit form.

1. Introduction

It is known that the Cauchy problem for elliptic equations is incorrect: the solution to the problem is unique, but unstable. The Cauchy problem for matrix factorizations of the Helmholtz equation, like many Cauchy problems for finding regular solutions of elliptic equations, in the general case is unstable with respect to uniformly small changes in the initial data. Thus, these tasks are incorrectly posed [1]. In unstable problems, the image of the operator is not closed, therefore, the solvability condition cannot be written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on a part of the boundary of a domain, the solution is usually unique, the problem is solvable for an everywhere dense data set, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than the theory of solvability of the Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg [2], A.M. Kytmanov and N.N. Tarkhanov [3]. In work [3], an integral formula was proved for systems of equations of elliptic type of the first order with constant coefficients in a bounded domain. For special domains, the problem of continuing limited analytic functions in the case when data is specified only on a part of the boundary was considered by T. Karleman [4]. The research of T. Karleman was continued by G.M. Goluzin and V.I. Krylov. The use of the classical Green formula for constructing a regularized solution of the Cauchy problem for the Laplace equation was proposed by academician M.M. Lavrent'ev in his famous monograph [5]. Using the ideas of M. M. Lavrent'ev [5,6], Sh. Yarmukhamedov constructed in explicit form a regularized solution of the Cauchy problem for the Laplace equation (see for instance [7]) The construction of the Carleman matrix for elliptic systems was carried out by Sh. Yarmukhamedov, N.N. Tarkhanov, A.A.

Shlapunov, I.E. Niyozov and others. In papers [8-20], the questions of exact and approximate solutions of the ill-posed Cauchy problem for various factorizations of the Helmholtz equations are studied. Such problems arise in mathematical physics and in various fields of natural science (for example, in electro-geological exploration, in cardiology, in electrodynamics, etc.)

2. Solution of the Cauchy problem on the plane

Let R^2 be a two-dimensional real Euclidean space, $x=(x_1,x_2)\in R^2$, $y=(y_1,y_2)\in R^2$.

$G\in R^2$ is a bounded simply connected domain with a piecewise smooth boundary consisting of the plane $T: y_2=0$ and some smooth curve S lying in the half-space $y_2>0$, i.e. $\partial G=S\bigcup T$.

We introduce the following notation:

$$r=|y-x|,\ \alpha=|y-x|,\ w=i\sqrt{u^2+\alpha^2}+y_2,\ u\ge 0,\ \partial_x=\left(\partial_{x_1},\partial_{x_2}\right)^T,\ \partial_x=\xi^T,$$

$$\xi^T=\left(\xi_1\ \xi_2\right)^T - \text{-transposed vektor } \xi,\ U(x)=(U_1(x),\dots,U_n(x))^T,\ u^0=(1,\dots,1)\in R^n,$$

$$n=2^m,\ m=2,\ E(z)=\begin{Vmatrix} z_1\dots 0 \\ \dots\dots \\ 0\dots z_n \end{Vmatrix} \text{ - diagonal matrix, } z=(z_1,\dots,z_n)\in R^n$$

Let $D(\xi^T)$, $(n\times n)$ - be a matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the condition is satisfied:

$$D^*(\xi^T)D(\xi^T)=E((|\xi|^2+\lambda^2)u^0) \tag{1}$$

where $D^*(\xi^T)$ - Hermitian conjugate matrix to $D(\xi^T)$, $|\xi|^2=\sum_{j=1}^{2}\xi_j^2$, $\lambda-$ real number.

Consider in the domain G a system of partial differential equations of the first order with constant coefficients of the form

$$D(\partial_x)U(x)=0, \tag{2}$$

where $D(\partial_x)$ is the matrix of differential operators of the first order.

We denote by $A(G)$ the class of vector functions in the domain G continuous on $\overline{G}=G\bigcup\partial G$ and satisfying system (2).

The Cauchy problem 1. Suppose $U(y)\in A(G)$ and

$$U(y)|_S=f(y),\ y\in S. \tag{3}$$

Here, $f(y)$ a given continuous vector-function on S.

It is required to restore the vector function $U(y)$ in the domain G, based on it's values $f(y)$ on S.

Example 1. Let given a system of first-order partial differential equations of the form (see, for instance [20])

$$\begin{cases} \partial_{x_1}U_1-\partial_{x_2}U_2+iU_4=0, \\ \partial_{x_2}U_1+\partial_{x_1}U_2+iU_3=0, \\ -\partial_{x_1}U_3+\partial_{x_1}U_4-iU_2=0, \\ \partial_{x_2}U_3+\partial_{x_1}U_4+iU_1=0. \end{cases}$$

Assuming $\partial_{x_1}\to\xi_1$, $\partial_{x_2}\to\xi_2$, we compose the following matrices:

$$D(\xi^T)=\begin{pmatrix} \xi_1 & \xi_2 & 0 & i \\ -\xi_2 & \xi_1 & -i & 0 \\ 0 & i & -\xi_1 & \xi_2 \\ i & 0 & \xi_2 & \xi_1 \end{pmatrix},\quad D^*(\xi^T)=\begin{pmatrix} \xi_1 & -\xi_2 & 0 & -i \\ \xi_2 & \xi_1 & -i & 0 \\ 0 & i & -\xi_1 & \xi_2 \\ -i & 0 & \xi_2 & \xi_1 \end{pmatrix}.$$

Relationship (1) is easily verified.

If G is a bounded and $U(y) \in A(G)$, then the following integral formula of Cauchy type is true

$$U(x) = \int_{\partial G} M(y,x) U(y) ds_y, \quad x \in G, \tag{4}$$

where

$$M(y,x) = \left(E\left(-\frac{i}{4} H_0^{(1)}(\lambda r) u^0 \right) D^*\left(\frac{\partial}{\partial y} \right) \right) D(t^T).$$

Here $t = (t_1, t_2)$ is the unit external normal, drawn at a point y, the curve ∂G, $-\frac{i}{4} H_0^{(1)}(\lambda r)$ is the fundamental solution of the Helmholtz equation in R^2.

We denote by $K(w)$ is an entire function taking real values for real w, ($w = u + iv,\ u, v -$ real numbers) and satisfying the following conditions:

$$K(u) \neq 0, \ \sup_{v \geq 1} \left| v^p K^{(p)}(w) \right| = \mathrm{M}(u,p) < \infty, \ -\infty < u < \infty, \ p = 0,1,2. \tag{5}$$

We define the function $\Phi(y, x)$ at $y \neq x$ by the following equality:

$$\Phi(y,x) = -\frac{1}{2\pi K(x_2)} \int_0^\infty \mathrm{Im} \frac{K(w)}{w - x_2} \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du. \tag{6}$$

Here $I_0(\lambda u) = J_0(i\lambda u)$ is the Bessel function of the first kind of zero order

Formula (4) is true if instead $-\frac{i}{4} H_0^{(1)}(\lambda r)$ of substituting the function

$$\Phi(y,x) = -\frac{i}{4} H_0^{(1)}(\lambda r) + g(y,x), \tag{7}$$

where $g(y, x)$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then the integral formula (4) has the following form

$$U(x) = \int_{\partial G} N(y,x) U(y) ds_y, \quad x \in G, \tag{8}$$

where

$$N(y,x) = \left(E\left(\Phi(y,x) u^0 \right) D^*\left(\frac{\partial}{\partial y} \right) \right) D(t^T).$$

Next, we use the following equalities:

$$2\pi K(x_2) \frac{\partial \Phi(y,x)}{\partial y_1} = \frac{(y_1 - x_1) \mathrm{Re}\, K(w_0) - sign(y_1 - x_1)(y_2 - x_2) \mathrm{Im}\, K(w_0)}{r^2} -$$

$$-(y_1 - x_1)\lambda \int_0^\infty \frac{\sqrt{u^2 + \alpha^2}\ \mathrm{Re}\, K(w) - (y_2 - x_2) \mathrm{Im}\, K(w)}{u^2 + r^2} \cdot \frac{I_1(\lambda u) du}{\sqrt{u^2 + \alpha^2}}, \ y \neq x, \tag{9}$$

$$w_0 = i|y_1 - x_1| + y_2, \ I_1(\lambda u) = I_0'(\lambda u),$$

and

$$2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_2}=\frac{(y_2-x_2)\operatorname{Re}K(w_0)-(y_1-x_1)\operatorname{Im}K(w_0)}{r^2}-$$
$$-\lambda\int_0^\infty\frac{(y_2-x_2)\operatorname{Re}K(w)-\sqrt{u^2+\alpha^2}\operatorname{Im}K(w)}{u^2+r^2}I_1(\lambda u)du,\ y_1\neq x_1, \tag{10}$$

which are obtained from (6).

By choosing the entire function $K(w)$ we obtain the following results:

In the formula (6) choosing

$$K(w)=\exp(\sigma w),\ K(x_2)=\exp(\sigma x_2),\ \sigma>0, \tag{11}$$

we get

$$\Phi_\sigma(y,x)=-\frac{e^{-\sigma x_2}}{2\pi}\int_0^\infty\operatorname{Im}\frac{\exp(\sigma w)}{w-x_2}\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du, \tag{12}$$
$$\sigma\geq\lambda+\sigma_0,\ \sigma_0>0.$$

Then the integral formula (8) has the following form:

$$U(x)=\int_{\partial G}N_\sigma(y,x)U(y)ds_y,\quad x\in G, \tag{13}$$

where

$$N_\sigma(y,x)=\left(E\left(\Phi_\sigma(y,x)u^0\right)D^*\left(\frac{\partial}{\partial y}\right)\right)D(t^T).$$

Theorem 1. *Let* $U(y)\in A(G)$ *it satisfy the inequality*

$$|U(y)|\leq 1,\ y\in T. \tag{14}$$

If

$$U_\sigma(x)=\int_S N_\sigma(y,x)U(y)ds_y,\ x\in G, \tag{15}$$

then the following estimate is true

$$|U(x)-U_\sigma(x)|\leq C(\lambda,x)\sigma e^{-\sigma x_2},\ \sigma>1,\ x\in G. \tag{16}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C(\lambda,x)$.

Proof. Using the integral formula (13) and the equality (15), we obtain

$$U(x)=U_\sigma(x)+\int_a^b N_\sigma(y,x)U(y)ds_y,\ x\in G.$$

Taking into account the inequality (14), we estimate the following

$$|U(x)-U_\sigma(x)|\leq\left|\int_T N_\sigma(y,x)U(y)ds_y\right|\leq$$
$$\int_T|N_\sigma(y,x)||U(y)|ds_y\leq\int_T|N_\sigma(y,x)|ds_y,\ x\in G. \tag{17}$$

We estimate the integrals $\int_a^b|\Phi_\sigma(y,x)|dy_1,\int_a^b\left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_1}\right|ds_y$ and $\int_a^b\left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_2}\right|ds_y$ on the part T of the plane $y_2=0$.

Let $\sigma>0$. Separating the imaginary part of equality (12), we obtain

$$\Phi_\sigma(y,x) = \frac{e^{\sigma(y_2-x_2)}}{2\pi}\left[\int_0^\infty \frac{\cos\sigma\sqrt{u^2+\alpha^2}}{u^2+r^2} uI_0(\lambda u)du - \right. \tag{18}$$

$$\left. - \int_0^\infty \frac{y_2 \sin\sigma\sqrt{u^2+\alpha^2}}{u^2+r^2} \frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}} du, \; y \neq x, \; x_2 > 0. \right.$$

Given (18) and the inequality

$$I_0(\lambda u) \le \exp(\lambda u), \tag{19}$$

we have

$$\int_a^b |\Phi_\sigma(y,x)| ds_y \le C(\lambda,x)\sigma e^{-\sigma x_2}, \; \sigma > 1, \; x \in G. \tag{20}$$

To estimate the integrals $\int_a^b \left|\frac{\partial \Phi_\sigma(y,x)}{\partial y_1}\right| ds_y$ and $\int_a^b \left|\frac{\partial \Phi_\sigma(y,x)}{\partial y_2}\right| ds_y$, we use equalities (9) and (10). To do this, using equalities (11) and choosing

$$K(w_0) = \exp(\sigma w_0), \; \sigma > 0, \tag{21}$$

we obtain the following formulas

$$2\pi e^{\sigma x_2} \frac{\partial \Phi_\sigma(y,x)}{\partial y_1} = \frac{(y_1-x_1)\operatorname{Re}\exp(\sigma w_0) + sign(y_1-x_1)(y_2-x_2)\operatorname{Im}\exp(\sigma w_0)}{r^2} - \tag{22}$$

$$-(y_1-x_1)\lambda\int_0^\infty \frac{\sqrt{u^2+\alpha^2}\operatorname{Re}\exp(\sigma w) - (y_2-x_2)\operatorname{Im}\exp(\sigma w)}{u^2+r^2} \cdot \frac{I_1(\lambda u)du}{\sqrt{u^2+\alpha^2}}, \; y \neq x,$$

and

$$2\pi e^{\sigma x_2} \frac{\partial \Phi_\sigma(y,x)}{\partial y_2} = \frac{(y_2-x_2)\operatorname{Re}\exp(\sigma w_0) + (y_1-x_1)\operatorname{Im}\exp(\sigma w_0)}{r^2} - \tag{23}$$

$$-\lambda\int_0^\infty \frac{(y_2-x_2)\operatorname{Re}\exp(\sigma w) - \sqrt{u^2+\alpha^2}\operatorname{Im}\exp(\sigma w)}{u^2+r^2} I_1(\lambda u)du, \; y_1 \neq x_1.$$

Given equality (22) and inequality

$$I_1(\lambda u) \le \lambda u \exp(\lambda u), \tag{24}$$

we get

$$\int_a^b \left|\frac{\partial \Phi_\sigma(y,x)}{\partial y_1}\right| ds_y \le C(\lambda,x)\sigma e^{-\sigma x_2}, \; \sigma > 1, \; x \in G. \tag{25}$$

Similarly, taking into account equality (2.16) and inequality (24), we estimate the following integral

$$\int_a^b \left|\frac{\partial \Phi_\sigma(y,x)}{\partial y_2}\right| ds_y \le C(\lambda,x)\sigma e^{-\sigma x_2}, \; \sigma > 1, \; x \in G. \tag{26}$$

From inequalities (20), (25) and (26) we obtain (16).

Theorem 1 is proved.

Corollary 1. *The limiting equality*

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x)$$

holds uniformly on each compact set in the domain G.

Theorem 2. *Let* $U(y)\in A(G)$ *satisfy condition (14) on a part of the plane* $y_2 = 0$, *and on a smooth curve* S *the inequality*

$$|U(y)| \le \delta,\ 0<\delta<1, \tag{27}$$

where $\bar{y}_2 = \max_{y\in S} y_2$.

Then the following estimate holds

$$|U(x)| \le C(\lambda,x)\sigma\delta^{\frac{x_2}{\bar{y}_2}},\ \sigma>1,\ x\in G. \tag{28}$$

Proof. From (13) and equality (15) as $x\in G$, we have

$$U(x) = \int_S N_\sigma(y,x)U(y)ds_y + \int_a^b N_\sigma(y,x)U(y)ds_y. \tag{29}$$

We estimate the following

$$|U(x)| \le \left|\int_S N_\sigma(y,x)U(y)ds_y\right| + \left|\int_a^b N_\sigma(y,x)U(y)ds_y\right|,\ x\in G. \tag{30}$$

Given inequality (27), we estimate the first term in inequality (30).

$$\left|\int_S N_\sigma(y,x)U(y)ds_y\right| \le \int_S |N_\sigma(y,x)||U(y)|ds_y \le$$
$$\le \delta\int_S |N_\sigma(y,x)|ds_y,\ x\in G. \tag{31}$$

We estimate the integrals $\int_S |\Phi_\sigma(y,x)|\,ds_y$, $\int_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_1}\right|ds_y$ and $\int_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_2}\right|ds_y$ on a smooth curve S.

Taking into account equality (18) and inequality (19), we have

$$\int_S |\Phi_\sigma(y,x)|\,ds_y \le C(\lambda,x)\sigma e^{\sigma(\bar{y}_2-x_2)},\ \sigma>1,\ x\in G. \tag{32}$$

Using equality (2.15) and inequality (2.17), we have

$$\int_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_1}\right|ds_y \le C(\lambda,x)\sigma e^{\sigma(\bar{y}_2-x_2)},\ \sigma>1,\ x\in G. \tag{33}$$

Similarly, using equality (23) and inequality (24), we obtain

$$\int_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_2}\right|ds_y \le C(\lambda,x)\sigma e^{\sigma(\bar{y}_2-x_2)},\ \sigma>1,\ x\in G. \tag{34}$$

From (32) - (34) we obtain

$$\left|\int_S N_\sigma(y,x)U(y)ds_y\right| \le C(\lambda,x)\sigma\delta e^{\sigma(\bar{y}_2-x_2)},\ \sigma>1,\ x\in G. \tag{35}$$

The following is known

$$\left|\int_a^b N_\sigma(y,x)U(y)ds_y\right| \le C(\lambda,x)\sigma e^{-\sigma x_2},\ \sigma>1,\ x\in G. \tag{36}$$

Now, taking into account (35) - (36), we have

$$|U(x)| \le \frac{C(\lambda,x)\sigma}{2}(\delta e^{\sigma\bar{y}_2}+1)e^{-\sigma x_2},\ \sigma>1,\ x\in G. \tag{37}$$

Choosing σ from the equality

$$\sigma = \frac{1}{\bar{y}_2}\ln\frac{1}{\delta}, \tag{38}$$

we obtain inequality (28).

Theorem 2 is proved.

Let $U(y)\in A(G)$ and instead $U(y)$ on S with its approximation $f_\delta(y)$, respectively, with an error, $0<\delta<1$,

$$\max_S |U(y)-f_\delta(y)| \le \delta.$$

We set

$$U_{\sigma(\delta)}(x) = \int_S N_\sigma(y,x)\,f_\delta(y)ds_y,\ x\in G. \tag{39}$$

The following is true.

Theorem 3. *Let $U(y)\in A(G)$ on the part of the plane $y_2=0$ satisfy condition (9).*

Then the following estimate holds.

$$\left|U(x)-U_{\sigma(\delta)}(x)\right| \le C(\lambda,x)\sigma\delta^{\frac{x_2}{\bar{y}_2}},\ \sigma>1,\ x\in G. \tag{40}$$

Proof. From the integral formulas (13) and equality (39), we have

$$U(x)-U_{\sigma(\delta)}(x) = \int_S N_\sigma(y,x)\{U(y)-f_\delta(y)\}ds_y + \int_a^b N_\sigma(y,x)U(y)ds_y.$$

Now, repeating the proofs of Theorems 1 and 2, we obtain

$$\left|U(x)-U_{\sigma(\delta)}(x)\right| \le \frac{C(\lambda,x)\sigma}{2}(\delta e^{\sigma\bar{y}_2}+1)e^{-\sigma x_2}.$$

Hence, choosing σ from equality (38), we obtain (40).

Theorem 3 is proved.

Corollary 2 *The limiting equality*

$$\lim_{\delta\to 0} U_{\sigma(\delta)}(x) = U(x)$$

holds uniformly on each compact set in the domain G.

Thus, the functional $U_{\sigma(\delta)}(x)$ is a regularization of the solution of the problem (2) - (3).

3. Solution of the Cauchy problem in space

Let R^3 be the three-dimensional real Euclidean space,

$$x=(x_1,x_2,x_3)\in R^3,\ y=(y_1,y_2,y_3)\in R^3,\ x'=(x_1,x_2)\in R^2,\ y'=(y_1,y_2)\in R^2.$$

$G \subset \mathrm{R}^3$ be a bounded simply-connected domain with piecewise smooth boundary consisting of the plane T : $y_3 = 0$ and of a smooth surface S lying in the half-space $y_3 > 0$, that i.s., $\partial G = S \bigcup T$.

We introduce the following notation:

$$r = |y - x|,\ \alpha = |y' - x'|,\ w = i\sqrt{u^2 + \alpha^2} + y_3,\ u \ge 0,\ \ \partial_x = \left(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}\right)^T,\ \ \partial_x \to \xi^T,$$

$$\xi^T = \left(\xi_1\ \xi_2\ \xi_3\right)^T - \text{transposed vector } \xi,\ U(x) = (U_1(x),...,U_n(x))^T,\ u^0 = (1,...,1) \in \mathrm{R}^n,$$

$$n = 2^m,\ m = 3,\ \ E(z) = \begin{Vmatrix} z_1 ... 0 \\ .... \\ 0 ... z_n \end{Vmatrix} - \text{diagonal matrix},\ z = (z_1,...,z_n) \in \mathrm{R}^n.$$

Let $D(x^T)$ the $(n \times n)-$the matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0), \tag{41}$$

where $D^*(x^T)$ is the Hermitian conjugate matrix $D(x^T)$, $|\xi|^2 = \sum_j^3 \xi_j^2$, $\lambda -$ real number

Consider in the region G a system of differential equations in partial derivatives of the first order

$$D(\partial_x)U(x) = 0, \tag{42}$$

where $D(\partial_x)$ is the matrix of differential operators of the first order.

We denote by $A(G)$ the class of vector functions in the domain G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (42).

The Cauchy problem 2. Suppose $U(y) \in A(G)$ and

$$U(y)\big|_S = f(y),\ y \in S. \tag{43}$$

Here, $f(y)$ a given continuous vector-function on S.

It is required to restore the vector function $U(y)$ in the domain G, based on it's values $f(y)$ on S.

Example 2. Let a system of first-order partial differential equations of the form

$$\begin{cases} \partial_{x_1}U_1 + \partial_{x_2}U_4 + \partial_{x_3}U_6 + iU_8 = 0, \\ \partial_{x_1}U_2 + \partial_{x_2}U_3 + \partial_{x_3}U_5 + iU_7 = 0, \\ \partial_{x_2}U_2 - \partial_{x_1}U_3 + \partial_{x_3}U_8 + iU_6 = 0, \\ -\partial_{x_2}U_1 + \partial_{x_1}U_4 + \partial_{x_3}U_7 + iU_5 = 0, \\ \partial_{x_3}U_2 + \partial_{x_1}U_5 + \partial_{x_2}U_8 + iU_4 = 0, \\ \partial_{x_3}U_1 - \partial_{x_1}U_6 + \partial_{x_2}U_7 + iU_3 = 0, \\ \partial_{x_3}U_4 - \partial_{x_2}U_6 + \partial_{x_3}U_7 + iU_2 = 0, \\ \partial_{x_3}U_3 + \partial_{x_2}U_5 + \partial_{x_1}U_8 + iU_1 = 0. \end{cases}$$

Assuming $\partial_{x_1} \to \xi_1$, $\partial_{x_2} \to \xi_2$ and $\partial_{x_3} \to \xi_3$, we obtain the matrices

$$D(\xi^T) = \begin{pmatrix} \xi_1 & 0 & 0 & \xi_2 & 0 & \xi_3 & 0 & i \\ 0 & \xi_1 & \xi_2 & 0 & \xi_3 & 0 & i & 0 \\ 0 & \xi_2 & -\xi_1 & 0 & 0 & i & 0 & \xi_3 \\ -\xi_2 & 0 & 0 & \xi_1 & i & 0 & \xi_3 & 0 \\ 0 & \xi_3 & 0 & i & \xi_1 & 0 & 0 & \xi_2 \\ \xi_3 & 0 & i & 0 & 0 & -\xi_1 & \xi_2 & 0 \\ 0 & i & 0 & \xi_3 & 0 & -\xi_2 & \xi_1 & 0 \\ i & 0 & \xi_3 & 0 & \xi_2 & 0 & 0 & \xi_1 \end{pmatrix},\ D^*(\xi^T) = \begin{pmatrix} \xi_1 & 0 & 0 & -\xi_2 & 0 & \xi_3 & 0 & -i \\ 0 & \xi_1 & \xi_2 & 0 & \xi_3 & 0 & -i & 0 \\ 0 & \xi_2 & -\xi_1 & 0 & 0 & -i & 0 & \xi_3 \\ \xi_2 & 0 & 0 & \xi_1 & -i & 0 & \xi_3 & 0 \\ 0 & \xi_3 & 0 & -i & \xi_1 & 0 & 0 & \xi_2 \\ \xi_3 & 0 & -i & 0 & 0 & -\xi_1 & -\xi_2 & 0 \\ 0 & -i & 0 & \xi_3 & 0 & \xi_2 & \xi_1 & 0 \\ -i & 0 & \xi_3 & 0 & \xi_2 & 0 & 0 & \xi_1 \end{pmatrix}$$

Relation (41) is easily verified.

If G is a bounded and $U(y) \in A(G)$, then the following integral formula of Cauchy type is true

$$U(x) = \int_{\partial G} M(y,x)U(y)ds_y, \quad x \in G, \tag{44}$$

where

$$M(y,x) = \left(E\left(-\frac{e^{i\lambda r}}{4\pi r} u^0 \right) D^*\left(\frac{\partial}{\partial y} \right) \right) D(t^T).$$

Here $t = (t_1, t_2, t_3)$ is the unit exterior normal, drawn at a point y, the surface ∂G, $-\frac{e^{i\lambda r}}{4\pi r}$ is the fundamental solution of the Helmholtz equation in R^3.

We denote by $K(w)$ is an entire function taking real values for real w ($w = u + iv;\, u, v -$real numbers) and satisfying the following conditions:

$$K(u) \neq 0, \sup_{v \geq 1} \left| v^p K^{(p)}(w) \right| = \mathrm{M}(u,p) < \infty, \; -\infty < u < \infty, \; p = 0,1,2,3. \tag{45}$$

We define a function $\Phi(y,x)$ when $y \neq x$ by the following equality:

$$\Phi(y,x) = -\frac{1}{2\pi^2 K(x_3)} \int_0^\infty \mathrm{Im} \frac{K(w)}{w - x_3} \frac{\cos \lambda u}{\sqrt{u^2 + \alpha^2}} du. \tag{46}$$

Formula (44) is true if instead of $-\frac{e^{i\lambda r}}{4\pi r}$ we substitute the function

$$\Phi(y,x) = -\frac{e^{i\lambda r}}{4\pi r} + g(y,x), \tag{47}$$

where $g_\sigma(y,x)$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then the integral formula (44) has the following form

$$U(x) = \int_{\partial G} N(y,x)U(y)ds_y, \quad x \in G, \tag{48}$$

where

$$N(y,x) = \left(E\left(\Phi(y,x) u^0 \right) D^*\left(\frac{\partial}{\partial y} \right) \right) D(t^T).$$

In the formula (1.40), choosing

$$K(w) = \exp(\sigma w), K(x_3) = \exp(\sigma x_3), \sigma > 0,$$

we get

$$\Phi_\sigma(y,x) = -\frac{e^{-\sigma x_3}}{2\pi^2} \int_0^\infty \mathrm{Im} \frac{\exp(\sigma w)}{w - x_3} \frac{\cos \lambda u}{\sqrt{u^2 + \alpha^2}} du, \tag{49}$$

Then the integral formula (48) has the form:

$$U(x) = \int_{\partial G} N_\sigma(y,x)U(y)ds_y, \quad x \in G, \tag{50}$$

where

$$N_\sigma(y,x) = \left(E\left(\Phi_\sigma(y,x) u^0 \right) D^*\left(\frac{\partial}{\partial y} \right) \right) D(t^T).$$

Theorem 4. *Let* $U(y) \in A(G)$ *it satisfy the inequality*

$$|U(y)| \le 1,\ y \in T. \tag{51}$$

If

$$U_\sigma(x) = \int_S N_\sigma(y,x) U(y) ds_y,\ x \in G, \tag{52}$$

then the following estimate is true

$$|U(x) - U_\sigma(x)| \le C(x)\sigma e^{-\sigma x_3},\ \sigma > 1,\ x \in G. \tag{53}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C(x)$.

Proof. Using the integral formula (50) and the equality (52), we obtain

$$U(x) = U_\sigma(x) + \int_T N_\sigma(y,x) U(y) ds_y,\ x \in G.$$

Taking into account the inequality (51), we estimate the following

$$\begin{aligned} |U(x) - U_\sigma(x)| &\le \left| \int_T N_\sigma(y,x) U(y) ds_y \right| \le \\ &\le \int_T |N_\sigma(y,x)||U(y)| ds_y \le \int_T |N_\sigma(y,x)| ds_y,\ x \in G. \end{aligned} \tag{54}$$

To do this, we estimate the integrals $\int_T |\Phi_\sigma(y,x)| ds_y$, $\int_T \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_j} \right| ds_y$, $j = 1,2$ and $\int_T \left| \frac{\partial \Phi_\sigma}{\partial y_3}(y,x) \right| ds_y$ on the part T of the plane $y_3 = 0$.

Separating the imaginary part of (49), we obtain

$$\begin{aligned} \Phi_\sigma(y,x) = \frac{e^{\sigma(y_3 - x_3)}}{2\pi^2} \Bigg[&\int_0^\infty \frac{\cos \sigma\sqrt{u^2+\alpha^2}}{u^2 + r^2} \cos \lambda u\, du - \\ &- \int_0^\infty \frac{(y_3 - x_3) \sin \sigma\sqrt{u^2+\alpha^2}}{u^2+r^2} \frac{\cos \lambda u}{\sqrt{u^2+\alpha^2}} du \Bigg],\ x_3 > 0. \end{aligned} \tag{55}$$

Taking into account equality (55), we have

$$\int_T |\Phi_\sigma(y,x)| ds_y \le C(x)\sigma e^{-\sigma x_3},\ \sigma > 1,\ x \in G. \tag{56}$$

To estimate the second integral, we use the equality

$$\frac{\partial \Phi_\sigma(y,x)}{\partial y_j} = \frac{\partial \Phi_\sigma(y,x)}{\partial s} \frac{\partial s}{\partial y_j} = 2(y_j - x_j) \frac{\partial \Phi_\sigma(y,x)}{\partial s},\ j = 1,2. \tag{57}$$

where

$$\begin{aligned} \frac{\partial \Phi_\sigma(y,x)}{\partial s} = \frac{e^{\sigma(y_3-x_3)}}{2\pi^2} \Bigg[&\int_0^\infty \left(\frac{-\sigma \sin \sigma\sqrt{u^2+\alpha^2}}{2(u^2+r^2)\sqrt{u^2+\alpha^2}} - \right. \\ &\left. - \frac{\cos \sigma\sqrt{u^2+\alpha^2}}{(u^2+r^2)^2} \right) \cos \lambda u\, du + \int_0^\infty \left(\frac{\sigma (y_3 - x_3) \cos \sigma\sqrt{u^2+\alpha^2}}{2(u^2+r^2)(u^2+\alpha^2)} - \right. \\ &\left. - \frac{(y_3-x_3)\sin \sigma\sqrt{u^2+\alpha^2}}{(u^2+r^2)^2\sqrt{u^2+\alpha^2}} - \frac{(y_3-x_3)\sin \sigma\sqrt{u^2+\alpha^2}}{2(u^2+r^2)(u^2+\alpha^2)^{3/2}} \right) \cos \lambda u\, du \Bigg],\ s = \alpha^2. \end{aligned} \tag{58}$$

Taking into account (57) - (58), we obtain

$$\int_T \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_1} \right| ds_y \le C(x)\sigma e^{-\sigma x_3},\ \sigma > 1,\ x \in G. \tag{59}$$

Similarly we obtain

$$\int_T \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_2} \right| ds_y \le C(x)\sigma e^{-\sigma x_3},\ \sigma > 1,\ x \in G. \tag{60}$$

To estimate the integral $\int_T \left| \frac{\partial \Phi_\sigma}{\partial y_3}(y,x) \right| ds_y$, we use the equality

$$\frac{\Phi_\sigma(y,x)}{\partial y_3} = \frac{e^{\sigma(y_3-x_3)}}{2\pi^2}\left[\int_0^\infty \left(\frac{\sigma\cos\sigma\sqrt{u^2+\alpha^2}}{u^2+r^2} - \right.\right.$$

$$\left. - \frac{2(y_3-x_3)\cos\sigma\sqrt{u^2+\alpha^2}}{(u^2+r^2)^2} \right) \cos\lambda u\, du - \int_0^\infty \left(\frac{\sigma(y_3-x_3)\sin\sigma\sqrt{u^2+\alpha^2}}{(u^2+r^2)\sqrt{u^2+\alpha^2}} - \right. \tag{61}$$

$$\left.\left. - \frac{\sin\sigma\sqrt{u^2+\alpha^2}}{(u^2+r^2)\sqrt{u^2+\alpha^2}} - \frac{2(y_3-x_3)^2\sin\sigma\sqrt{u^2+\alpha^2}}{(u^2+r^2)^2\sqrt{u^2+\alpha^2}} \right) \cos\lambda u\, du \right].$$

Taking into account the equality (61), we obtain

$$\int_T \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_3} \right| ds_y \le C(x)\sigma e^{-\sigma x_3},\ \sigma > 1,\ x \in G, \tag{62}$$

From the inequalities (56), (59), (60), and (62), we obtain (53).

Theorem 4 is proved.

Corollary 3. *The limiting equality*

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x)$$

holds uniformly on each compact set in the domain G.

Theorem 5. *Let* $U(y) \in A(G)$ *it satisfy condition (51), and on a smooth surface* S *the inequality*

$$|U(y)| \le \delta,\ 0 < \delta < 1, \tag{63}$$

where $\bar{y}_3 = \max_{y\in S} y_3$.

Then the following estimate is true

$$|U(x)| \le C(x)\sigma\delta^{\frac{x_3}{\bar{y}_3}},\ \sigma > 1,\ x \in G. \tag{64}$$

Proof. Using the integral formula (50), we have

$$U(x) = \int_S N_\sigma(y,x)U(y)ds_y + \int_T N_\sigma(y,x)U(y)ds_y,\ x \in G.$$

We estimate the following

$$|U(x)| \le \left| \int_S N_\sigma(y,x)U(y)ds_y \right| + \left| \int_T N_\sigma(y,x)U(y)ds_y \right|,\ x \in G. \tag{65}$$

Taking inequality (57) into account, we estimate the first integral in (65).

$$\left|\int_S N_\sigma(y,x)U(y)ds_y\right| \le \int_S |N_\sigma(y,x)||U(y)|ds_y \le \delta\int_S |N_\sigma(y,x)|ds_y,\ x\in G. \quad (66)$$

To do this, we estimate the integrals $\int_S |\Phi_\sigma(y,x)|\,ds_y$, $\int_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_j}\right|ds_y$, $j=1,2$ and $\int_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_3}\right|ds_y$ on a smooth surface S.

Taking into account the equality (55), we have

$$\int_S |\Phi_\sigma(y,x)|\,ds_y \le C(x)\sigma e^{\sigma(y_3-x_3)},\ \sigma>1,\ x\in G. \quad (67)$$

To estimate the second integral, we use equalities (3.10) and (3.11).

$$\int_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_1}\right|ds_y \le C(x)\sigma e^{\sigma(y_3-x_3)},\ \sigma>1,\ x\in G. \quad (68)$$

Similarly, using equalities (57) and (58) we obtain

$$\int_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_2}\right|ds_y \le C(x)\sigma e^{\sigma(y_3-x_3)},\ \sigma>1,\ x\in G. \quad (69)$$

Taking into account the equality (3.14), we obtain

$$\int_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_3}\right|ds_y \le C(x)\sigma e^{\sigma(y_3-x_3)},\ \sigma>1,\ x\in G. \quad (70)$$

From (67) - (70), we obtain

$$\left|\int_S N_\sigma(y,x)U(y)ds_y\right| \le C(x)\sigma\delta e^{\sigma(y_3-x_3)},\ \sigma>1,\ x\in G. \quad (71)$$

The following is known

$$\left|\int_T N_\sigma(y,x)U(y)ds_y\right| \le C(x)\sigma e^{-\sigma x_3},\ \sigma>1,\ x\in G. \quad (72)$$

Now taking into account (71) - (72), we have

$$|U(x)| \le \frac{C(x)\sigma}{2}(\delta e^{\sigma\bar{y}_3}+1)e^{-\sigma x_3},\ \sigma>1,\ x\in G. \quad (73)$$

Choosing σ from the equality

$$\sigma = \frac{1}{\bar{y}_3}\ln\frac{1}{\delta}, \quad (74)$$

we obtain the inequality (64).

Theorem 5 is proved.

Let $U(y)\in A(G)$ and instead $U(y)$ on S with its approximation $f_\delta(y)$, respectively, with an error, $0<\delta<1$,

$$\max_S |U(y)-f_\delta(y)| \le \delta.$$

We put

$$U_{\sigma(\delta)}(x) = \int_S N_\sigma(y,x)f_\delta(y)ds_y,\ x\in G. \quad (75)$$

The following is true

Theorem 6. *Let* $U(y) \in A(G)$ *on the part of the plane* $y_3 = 0$ *satisfy condition (51). Then the following estimate is true*

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \le C(x)\sigma\delta^{\frac{x_3}{\bar{y}_3}},\ \sigma > 1,\ x \in G. \tag{76}$$

Proof. From the integral formulas (.3) and (75), we have

$$U(x) - U_{\sigma(\delta)}(x) =$$
$$= \int_S N_\sigma(y,x)\{U(y) - f_\delta(y)\}ds_y + \int_S N_\sigma(y,x)U(y)ds_y.$$

Now, repeating the proof of Theorems 4 and 5, we obtain

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \le \frac{C(x)\sigma}{2}(\delta e^{\sigma\bar{y}_3} + 1)e^{-\sigma x_3}.$$

Hence, choosing σ from (74), we obtain (76).

Theorem 6 is proved.

Corollary 4. *The limiting equality*

$$\lim_{\delta \to 0} U_{\sigma(\delta)}(x) = U(x)$$

holds uniformly on each compact set in the domain G.

Thus, the functional $U_{\sigma(\delta)}(x)$ is a regularization of the solution of the problem (42) - (43).

4. Conclusion

In this work, on the basis of the Carleman matrix, explicitly constructed regularized solutions for matrix factorizations of the Helmholtz equation in two-dimensional and three-dimensional bounded domains. The resulting formula is an analogue of the classical formula of Riemann, Voltaire and Hadamard, which they constructed to solve the Cauchy problem in the theory of hyperbolic equations. An estimate of the stability of the solution of the Cauchy problem in the classical sense for matrix factorizations of the Helmholtz equation was presented. This problem can be considered when, instead of the exact data of the Cauchy problem we have their approximations with a given deviation in the uniform metric and under the assumption that the solution of the Cauchy problem is bounded on part T, of the boundary of the domain G. The regularized method defines a stable method for an approximate solution.

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Author contributions

Davron Aslonqulovich Juraev: Conceptualization, Methodology, Software **Mahir Jalal Oglu Jalalov:** Data curation, Writing-Original draft preparation, Software, Validation. **Vagif Rza Oglu Ibrahimov:** Visualization, Investigation, Writing-Reviewing and Editing.

Conflicts of interest

The authors declare no conflicts of interest.

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