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Fundamental solution for the Helmholtz equation

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Abstract
This paper deals with the construction of a family of fundamental solutions of the Helmholtz equation, parameterized by an entire function with certain properties. Functions that possess these properties are called the Carleman function. On the basis of the proved lemmas for the Helmholtz equation in two-dimensional and three-dimensional bounded domains, in what follows we will find a regularized solution of the Cauchy problem already for a multidimensional domain.

1. Introduction

In this paper, the main attention is paid to the construction of the Helmholtz equation in flat domains and domains of three-dimensional Euclidean space. As is known, fundamental solutions play an important role in the study of partial differential equations. The Helmholtz operator has an explicit fundamental solution of superexact type in the entire space. In 1926, T. Carleman [1] constructed a formula that connects the values of the analytic function of a complex variable at the points of the region with its values on a piece of the boundary of this region. The construction of the Carleman function makes it possible in these problems to construct a regularization and obtain an estimate of the conditional stability. It is known that the Helmholtz equation in different spaces has a fundamentally different solution. In the future, using the construction of constructing a fundamental solution, we will construct an approximate solution for the Helmholtz equation [2]. MM. Lavrent'ev, in his works on the Cauchy problem for the Laplace equation and for some other ill-posed problems of mathematical physics, indicated a method for distinguishing the correctness class and developed stable methods for solving them [3,4]. M.M. Lavrent'ev proposed the construction of a regularized solution of the Cauchy problem for the Laplace equation using the Carleman function.

Moreover, in the 1977s, Sh. Yarmukhamedov pointed out the construction of a family of fundamental solutions parametrized by an entire function with certain properties [5]. This construction is used to construct explicit formulas that restore solutions of elliptic equations in a domain from their Cauchy data on a piece of the domain boundary. Such formulas are also called Carleman formulas. The Carleman function for the Laplace equation was constructed by Sh. Yarmukhamedov [5] when part of the boundary is the surface of a cone, and A. A. Shlapunov [6], when part of the boundary is the surface of a sphere. The multidimensional Carleman formula was constructed by L.A. Aizenberg [7]. An analogue of the Carleman formula for one class of elliptic systems with constant coefficients on the plane is considered in the work of E.V. Arbuzov and A.L. Bukhgeim [8].

At present, interest in the so-called ill-posed problems has noticeably increased in the natural sciences. The concept of a correct problem was given by J. Hadamard and developed by A.N. Tikhonov at the beginning of the 20th century. A well-posed, or well-posed, problem is understood as a problem, the solution of which:

1) exists;
2) only;
3) sustainable.

Accordingly, the problem is ill-posed or, what is the same, ill-posed if its solution does not satisfy at least one of the above three conditions [9-12].

After constructing the Carleman function in ill-posed problems, one will have to approximately find a regularized solution. In addition, we note that most of the tasks that arise in practice are incorrect in the sense of the Hadamard–Tikhonov definition. After all, contrary to popular belief, even a problem with contradictory data can be solved, it is only necessary to reformulate its formulation in a special way. This question, along with many others, is studied in the theory of ill-posed problems. There was an opinion that ill-posed problems could not be encountered in solving physical and technical problems and that for ill-posed problems it was impossible to construct an approximate solution in the absence of stability. The expansion of automation tools in obtaining experimental data has led to a large increase in the volume of such data; the need to establish information about natural-science objects from them required the consideration of ill-posed problems. The development of electronic computing technology and its application to solving mathematical problems has changed the point of view on the possibility of constructing approximate solutions to ill-posed problems. The Carleman formula for various elliptic equations is built in works [13-17]. Ill-posed problems may arise in the processing of geophysical, geological, astronomical observations, in solving problems of optimal control and planning.

Using the construction of previous works, we prove the validity of the fundamental solution for the Helmholtz equation in the plane case. For the matrix factorization of the Helmholtz equation, the validity of fundamental solutions in various spaces was considered by the author [18-25].

2. Fundamental solution of the Helmholtz Equation on the two-dimensional plane

This section deals with the construction of a family of fundamental solutions of the Helmholtz equation, parameterized by an entire function with certain properties.

Let R^2 be a two-dimensional real Euclidean space,

$$x=(x_1,x_2)\in R^2,\ y=(y_1,y_2)\in R^2,\ \alpha=|y_1-x_1|,\ r=|y-x|.$$

$G\subset R^2$ is a bounded simply connected region whose boundary consists of a smooth curve $S=\partial G,\ \overline{G}=S\bigcup G$.

We consider the Helmholtz equation

$$\Delta U(y)+\lambda^2 U(y)=0, \tag{1}$$

where $\lambda>0$, $\Delta-$ is the Laplace operator.

We denote by $K(w)$ is an entire function taking real values for real w ($w=u+iv; u,v-$ real numbers) and satisfying the following conditions:

$$K(u)\neq 0,\ \sup_{v\geq 1}\left|v^p K^{(p)}(w)\right|=\mathrm{M}(u,p)<\infty,\ -\infty<u<\infty,\ p=0,1,2. \tag{2}$$

We define a function $\Phi(y,x)$ when $y\neq x$ by the following equality:

$$\Phi(y,x)=-\frac{1}{2\pi K(x_2)}\int_0^\infty \mathrm{Im}\frac{K(w)}{w-x_2}\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du,\ \ w=i\sqrt{u^2+\alpha^2}+y_2, \tag{3}$$

where $I_0(\lambda u)-$ is the Bessel function of the first kind of zero order.

Lemma 1. *The function* $\Phi(y,x)$ *can be represented as*

$$\Phi(y,x)=-\frac{i}{4}H_0^{(1)}(\lambda r)+g(y,x). \tag{4}$$

Here $-\frac{i}{4}H_0^{(1)}(\lambda r)-$ is the fundamental solution of the Helmholtz equation in P^2, defined through the Hankel function of the first kind, $g(y,x)-$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y=x$.

We note that the proof of the lemma remains valid if, in (3), for $K(w)$ we take an analytic function that is regular in some domain and takes real values for real w, satisfying condition (2).

Proof. For convenience, we introduce the notation

$$f(w)=\frac{K(w)}{w-x_2},\ \ w=i\sqrt{u^2+\alpha^2}+y_2,$$
$$\varphi(y,x)=\int_0^\infty f(w)\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du. \tag{5}$$

In these notation

$$\Phi(y,x)=\frac{1}{2\pi K(x_2)}\operatorname{Im}\varphi(y,x). \tag{6}$$

Our goal is to prove that the function $\varphi(y,x)$ is a solution of equation (1) with respect to the variable y at $\alpha>0$.

This will follow that the function $\Phi(y,x)$ is a solution of equation (1) with respect to y at $\alpha>0$.

Taking into account conditions (2), from formula (5), by differentiation we obtain

$$\frac{\partial\varphi(y,x)}{\partial y_1}=\int_0^\infty\frac{\alpha uif'(w)}{u^2+\alpha^2}I_0(\lambda u)du-\int_0^\infty\frac{\alpha uf(w)}{\left(u^2+\alpha^2\right)^{3/2}}I_0(\lambda u)du,\ \ y_1>x_1. \tag{7}$$

The first integral is integrable by parts

$$\int_0^\infty\frac{\alpha uif'(w)}{u^2+\alpha^2}I_0\left(\lambda u\right)du=\int_0^\infty\frac{\alpha I_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du=$$

$$=-f(w)+\int_0^\infty\frac{\alpha uI_0(\lambda u)}{\left(u^2+\alpha^2\right)^{3/2}}f(w)du-\lambda\int_0^\infty\frac{\alpha I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f(w)du.$$

Substituting these expressions in (7), we obtain

$$\frac{\partial\varphi(y,x)}{\partial y_1}=-f(w)-\lambda\int_0^\infty\frac{\alpha I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f(w)du,\ \ y_1>x_1. \tag{8}$$

In the same way, we get

$$\frac{\partial\varphi(y,x)}{\partial y_1}=f(w)-\lambda\int_0^\infty\frac{\alpha I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f(w)du,\ \ y_1<x_1. \tag{9}$$

Taking into account (8) and (9), we have

$$\frac{\partial\varphi^2(y,x)}{\partial y_1^2}=-if(w)-\lambda\int_0^\infty\frac{I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f(w)du+$$

$$+\int_0^\infty\frac{\alpha^2I_0'(\lambda u)}{\left(u^2+\alpha^2\right)^{3/2}}f(w)du-\lambda\int_0^\infty\frac{i\alpha^2I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}}f'(w)du,\ \ y_1\neq x_1,$$

or

$$\frac{\partial\varphi^2(y,x)}{\partial y_1^2} = -if'(w) - \lambda\int_0^\infty \frac{u^2 I_0'(\lambda u)}{\left(u^2+\alpha^2\right)^{3/2}} f(w)du - \int_0^\infty \frac{i\alpha^2 I_0'(\lambda u)}{\sqrt{u^2+\alpha^2}} f'(w)\,du. \tag{10}$$

Now we will calculate the partial derivatives of the function $\varphi(y,x)$ with respect to y_2 at $y_1 \neq x_1$.

$$\frac{\varphi(y,x)}{\partial y_2} = \int_0^\infty \frac{f'(w)\,u\,I_0(\lambda u)}{\sqrt{u^2+\alpha^2}} f(w)du, \quad y_1 \neq x_1.$$

Integrating in parts, we obtain

$$\frac{\partial\varphi(y,x)}{\partial y_2} = f(w) + i\lambda\int_0^\infty I_0'(\lambda u)\, f(w)du,\ \ y_1 \neq x_1. \tag{11}$$

From here

$$\frac{\partial^2\varphi(y,x)}{\partial y_2^2} = if'(w) + i\lambda\int_0^\infty I_0'(\lambda u)\, f'(w)\,du,\ \ y_1 \neq x_1. \tag{12}$$

Taking into account (10), (12) and (1), we have

$$\Delta\varphi(y,x) + \lambda^2\varphi(y,x) = -\lambda\int_0^\infty \frac{u^2 I_0'(\lambda u)}{\left(u^2+\alpha^2\right)^{3/2}} f(w)du +$$

$$+i\lambda\int_0^\infty \frac{u^2 f'(w)\, I_0'(\lambda u)}{u^2+\alpha^2}du + \lambda^2\int_0^\infty \frac{f(w)\, I_0'(\lambda u) u}{\sqrt{u^2+\alpha^2}}du,\quad y_1 \neq x_1.$$

Integrating the second integral in parts, we obtain

$$\Delta\varphi(y,x) + \lambda^2\varphi(y,x) = -\lambda\int_0^\infty \frac{f(w)}{\sqrt{u^2+\alpha^2}}\left[\lambda\, u I_0''(\lambda u) + I_0'(\lambda u) + \lambda\, u I_0(\lambda u)\right]du.$$

Since, the integrand

$$\lambda\, u I_0''(\lambda u) + I_0'(\lambda u) + \lambda\, u I_0(\lambda u) = 0$$

is the zero order Bessel equation and $I_0(\lambda u)$ is its solution, then

$$\Delta\varphi(y,x) + \lambda^2\varphi(y,x) = 0,\ \ y_1 \neq x_1.$$

It follows from this equality that $\Phi(y,x)$ is a solution of equation (1) with respect to y on the line $y_1 \neq x_1$. For this it is enough to show its differentiability as $y_1 = x_1$ (then, according to the well-known property of solving an elliptic equation, it continues on the line $y_1 = x_1$ as a solution).

Taking into account (6) and (11), we have

$$2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_2} = \operatorname{Re}\frac{K(i\alpha+y_2)}{i\alpha+y_2-x_2} + \lambda\int_0^\infty \operatorname{Re}\frac{K(w)}{w-x_2} I_1(\lambda u)du,$$

$$y_1 \neq x_1,\ \ I_0'(\lambda u) = -I_1(\lambda u),$$

or

$$2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_2} = \frac{(y_2-x_2)\operatorname{Re}K(w_0) - |y_1-x_1|\operatorname{Im}K(w_0)}{r^2} -$$

$$-\lambda\int_0^\infty \frac{(y_2-x_2)\operatorname{Re}K(w) - \sqrt{u^2+\alpha^2}\ \operatorname{Im}K(w)}{u^2+r^2} I_1(\lambda u)du,\ \ y_1 \neq x_1. \tag{13}$$

Since $I_1(\lambda u) \approx \sqrt{\frac{2}{\pi\lambda u}}$, $t \to \infty, t = \lambda u$ is true, then, taking into account condition (2), we see that $\frac{\partial\Phi(y,x)}{\partial y_2}$ is continuous on the line $y_1 = x_1$. For $\frac{\partial\Phi(y,x)}{\partial y_1}$ with $y_1 > x_1$, from (6) and (9) we have

$$2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_1} = -\operatorname{Im} f(i\alpha + y_2) + \lambda\int_0^\infty f(w)\frac{\alpha I_1(\lambda u)}{i\sqrt{u^2+\alpha^2}}du =$$

$$= \operatorname{Im}\frac{K(i\alpha + y_2)}{i\alpha + y_2 - x_2} + \lambda\int_0^\infty \operatorname{Im}\frac{K(w)}{w - x_2}\frac{\alpha I_1(\lambda u)}{\sqrt{u^2+\alpha^2}}du,$$

or

$$2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_1} = \frac{(y_1 - x_1)\operatorname{Re}K(w_0) - (y_2 - x_2)\operatorname{Im}K(w_0)}{r^2} -$$

$$-\lambda\int_0^\infty \frac{\sqrt{u^2+\alpha^2}\operatorname{Re}K(w) - (y_2 - x_2)\operatorname{Im}K(w)}{u^2 + r^2}\frac{(y_1 - x_1)I_1(\lambda u)}{\sqrt{u^2+\alpha^2}}du, \quad y_1 > x_1,$$

$$w_0 = i\alpha + y_2, \quad w = i\sqrt{u^2+\alpha^2} + y_2.$$

Similarly, from (9) we find

$$2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_1} = \frac{(y_1 - x_1)\operatorname{Re}K(w_0) - (y_2 - x_2)\operatorname{Im}K(w_0)}{r^2} -$$

$$-\lambda\int_0^\infty \frac{\sqrt{u^2+\alpha^2}\operatorname{Re}K(w) - (y_2 - x_2)\operatorname{Im}K(w)}{u^2 + r^2}\frac{(y_1 - x_1)I_1(\lambda u)}{\sqrt{u^2+\alpha^2}}du, \quad y_1 < x_1.$$

Combining the obtained formulas, we have

$$2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_1} = \frac{(y_1 - x_1)\operatorname{Re}K(w_0) - sign(y_1 - x_1)(y_2 - x_2)\operatorname{Im}K(w_0)}{r^2} -$$
$$-\lambda(y_1 - x_1)\int_0^\infty \frac{\sqrt{u^2+\alpha^2}\operatorname{Re}K(w) - (y_2 - x_2)\operatorname{Im}K(w)}{u^2 + r^2}\frac{I_1(\lambda u)}{\sqrt{u^2+\alpha^2}}du, \quad y \neq x. \tag{14}$$

Since $I_1(\lambda u) \approx \frac{\lambda u}{2}$ is true at $t \le 1$ and $I_1(\lambda u) \approx \sqrt{\frac{2}{\pi\lambda u}}$ is at $t \to \infty, t = \lambda u$, and the integral expresses a continuous function, it means that $\frac{\partial\Phi(y,x)}{\partial y_1}$ is continuous on the line $y_1 = x_1$.

From (3) for the sum

$$g(y,x) = \Phi(y,x) + \frac{i}{4}H_0^{(1)}(\lambda r),$$

we find

$$g(y,x) = -\frac{1}{2\pi K(x_2)}\int_0^\infty \mathrm{Im}\left[\frac{K(w)}{w-x_2}\right]\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du + \frac{i}{4}H_0^{(1)}(\lambda r) =$$

$$= -\frac{1}{2\pi K(x_2)}\int_0^1 \mathrm{Im}\left[\frac{K(w)-K(x_2)}{w-x_2}\right]\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du -$$

$$-\frac{1}{2\pi K(x_2)}\int_1^\infty \mathrm{Im}\left[\frac{K(w)}{w-x_2}\right]\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du -$$

$$-\frac{1}{2\pi}\int_0^1 \mathrm{Im}\left[\frac{1}{w-x_2}\right]\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du + \frac{i}{4}H_0^{(1)}(\lambda r).$$

The second and third terms depict the continuous function of a point y in R^2, including the point $y = x$ (integrands are continuous and the integrals converge uniformly with respect to $x \in R^2$). This follows from condition (1.8) and the asymptotes of the Bessel function $I_1(\lambda u) \approx \sqrt{\frac{2}{\pi\lambda u}}$ for $t \to \infty,\ t = \lambda u$.

The continuity of the first integral follows from the continuity of the integrand as a function of the point $(u, y),\ u \in [0;1],\ y \in R^2$.

Thus, the function $g(y,x)$ is a solution of equation (1) in $R^2 \setminus \{x\}$ and is continuous in R^2. According to the well-known property (on continuation) of solving an equation of elliptic type, it will be a solution in the variable y everywhere. The lemma1 is proved.

3. Fundamental solution of the Helmholtz Equation in a three-dimensional bounded domain

Now, consider a family of fundamental solutions of the Helmholtz equation in R^3.

Let $R^3 -$ be a three-dimensional real Euclidean space,

$$x = (x_1, x_2, x_3) \in R^3,\ y = (y_1, y_2, y_3) \in R^3,\ \alpha^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2,$$
$$r^2 = |y - x|^2 = \alpha^2 + (y_3 - x_3)^2,\ s = \alpha^2.$$

$G \subset R^3 -$ is a bounded simply connected region whose boundary consists of a smooth surface $S = \partial G,\ \overline{G} = S \cup G$.

We consider the Helmholtz equation

$$\Delta U(y) + \lambda^2 U(y) = 0, \tag{15}$$

where $\lambda > 0$, $\Delta -$ is the Laplace operator.

We denote by $K(w),\ w = u + iv -$ an entire function taking real values for real w ($u, v -$ are real numbers) and satisfying the conditions:

$$K(u) \neq 0,\ \sup_{v \geq 1}\left|v^p K^{(p)}(w)\right| = \mathrm{M}(u, p) < \infty,$$
$$-\infty < u < \infty,\quad p = 0,1,2,3. \tag{16}$$

The function $\Phi(y,x)$ at $y \neq x$ is defined by the following equality:

$$-2\pi^2 K(x_3)\Phi(y,x) = \int_0^\infty \mathrm{Im}\frac{K(i\sqrt{u^2+\alpha^2}+y_3)}{i\sqrt{u^2+\alpha^2}+y_3-x_3}\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}du,$$
$$w = i\sqrt{u^2+\alpha^2} + y_3. \tag{17}$$

Separating the imaginary part of equality (17), we obtain

$$-2\pi^2 K(x_3)\Phi(y,x)=\int_0^\infty \operatorname{Im}\frac{[\operatorname{Re}K(w)+i\operatorname{Im}K(w)](\overline{w}-x_3)}{(w-x_3)(\overline{w}-x_3)}\cdot\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}\,du=$$

$$=\int_0^\infty \frac{-\sqrt{u^2+\alpha^2}\operatorname{Re}K(w)+(y_3-x_3)\operatorname{Im}K(w)}{u^2+r^2}\cdot\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}\,du,\quad \overline{w}=-i\sqrt{u^2+\alpha^2}+y_3.$$

We have

$$-2\pi^2 K(x_3)\Phi(x,y)=\int_0^\infty F(x,y,u)\cos\lambda u\,du, \tag{18}$$

where

$$F(x,y,u)=\frac{\varphi(x,y,u)}{u^2+r^2},$$

$$\varphi(x,y,u)=-\operatorname{Re}K(w)+\frac{(y_3-x_3)\operatorname{Im}K(w)}{\sqrt{u^2+\alpha^2}}. \tag{19}$$

Lemma 2. *The function $\Phi(y,x)$, defined for $y\neq x$ by equalities (17) and (18), can be represented in the form:*

$$\Phi(y,x)=-\frac{e^{i\lambda r}}{4\pi r}+g(y,x), \tag{20}$$

where $g(y,x)-$ is a regular solution of the Helmholtz equation in R^3.

Proof. We divide the integration interval in (1.23) into two parts: $(0;\infty)=(0;1)\bigcup(1;\infty)$. Then we transform $g(y,x)$ in the form

$$g(y,x)=-\frac{1}{2\pi^2K(x_3)}\int_0^\infty \operatorname{Im}\left[\frac{K(w)}{w-x_3}\right]\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}\,du+\frac{e^{i\lambda r}}{4\pi r}=$$

$$=-\frac{1}{2\pi^2K(x_3)}\int_0^1 \operatorname{Im}\left[\frac{K(w)-K(x_3)}{w-x_3}\right]\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}\,du-$$

$$-\frac{1}{2\pi^2K(x_3)}\int_1^\infty \operatorname{Im}\left[\frac{K(w)}{w-x_3}\right]\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}\,du-$$

$$-\frac{1}{2\pi^2}\int_0^1 \operatorname{Im}\left[\frac{1}{w-x_3}\right]\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}\,du+\frac{e^{i\lambda r}}{4\pi r}.$$

The second integral is denoted by $g_1(y,x)$. Since $1\le u<\infty$, the function $g_1(y,x)$ belongs to the class $C^2(\overline{G})$ with respect to the variable y, including the point $y=x$.

We denote the first integral by $g_2(y,x)$. We transform this integral as follows:

$$g_2(y,x)=\frac{1}{2\pi^2K(x_3)}\int_0^1 \operatorname{Im}\left[\frac{K(w)-K(x_3)}{w-x_3}\right]\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}\,du+$$

$$+\frac{1}{2\pi^2}\int_0^1 \operatorname{Im}\left[\frac{1}{w-x_3}\right]\frac{\cos\lambda u}{\sqrt{u^2+\alpha^2}}\,du,$$

where

$$\mathrm{Im}\left[\frac{1}{w-x_3}\right]=-\frac{\sqrt{u^2+\alpha^2}}{u^2+r^2}.$$

Since

$$K_1(w)=\frac{K(w)-K(x_3)}{w-x_3}$$

is an entire function real for real w, then the decomposition

$$\frac{1}{\sqrt{u^2+\alpha^2}}\mathrm{Im}[K_1(w)]=\sum_{n=0}^{\infty}(-1)^n\frac{K_1^{(2n+1)}(y_3)}{(2n+1)!}(u^2+\alpha^2)^n$$

it follows that the first term is a function of class $C^2(\overline{G})$.

We denote by $\psi(x,y)$ the second integral of $g_2(y,x)$. Taking into account the equality

$$\cos u=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n}}{(2n)!},$$

we transform it as follows:

$$\psi(x,y)=-\int_0^1\frac{\cos\lambda u}{u^2+r^2}du=-\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^{2n}}{(2n)!}\int_0^1\frac{u^{2n}}{u^2+r^2}du=$$

$$=-\int_0^1\frac{du}{u^2+r^2}+\frac{\lambda^2}{2}\int_0^1\frac{u^2du}{u^2+r^2}-\sum_{n=2}^{\infty}\frac{(-1)^n\lambda^{2n}}{(2n)!}\int_0^1\frac{u^{2n}}{u^2+r^2}du.$$

The first integral is equal to

$$\int_0^1\frac{du}{u^2+r^2}=\frac{\pi}{2r}-\int_1^{\infty}\frac{du}{u^2+r^2}.$$

The second integral is equal to

$$\frac{\lambda^2}{2}\int_0^1\frac{u^2du}{u^2+r^2}=\frac{\lambda^2}{2}-\frac{r^2\lambda^2}{2}\left(\frac{\pi}{2r}-\int_1^{\infty}\frac{du}{u^2+r^2}\right).$$

Therefore, the function $\psi(x,y)$ has the form

$$\psi(x,y)=-\frac{\pi}{2r}+\frac{\lambda^2}{2}-\frac{\pi r\lambda^2}{4}+\left(1+\frac{r^2\lambda^2}{2}\right)\int_1^{\infty}\frac{du}{u^2+r^2}-\sum_{n=2}^{\infty}\frac{(-1)^n\lambda^{2n}}{(2n)!}\int_0^1\frac{u^{2n}}{u^2+r^2}du.$$

As

$$-\frac{\pi}{2r}-\frac{\pi r\lambda^2}{4}=-\frac{\pi}{2}\frac{e^{i\lambda r}}{r}-\frac{\pi r\lambda^2}{2}-\frac{i\pi\lambda}{2}+\frac{\pi}{2}\sum_{n=3}^{\infty}\frac{i^n\lambda^n r^{n-1}}{n!},$$

we get

$$g_2(y,x)=-\frac{\pi}{2}\frac{e^{i\lambda r}}{r}+g_3(y,x),$$

where $g_3(y,x)$ is a function from the class $C^2(\overline{G})$ with respect to the variable y, including the point $y=x$.

Assuming

$$g(x,y)=-\frac{g_1(y,x)+g_2(y,x)}{2\pi^2},$$

we obtain the statement of the lemma. The lemma 2 is proved.

Lemma 3. *For $y \neq x$, the following equality holds:*

$$\frac{1}{2\pi^2}\int_0^\infty \Delta_y F(x,y,u)\cos\lambda u\, du = \lambda^2 \Phi(x,y). \tag{21}$$

Proof. We denote

$$\psi(w) = -\frac{1}{2\pi^2 K(x_3)}\frac{K(w)}{w-x_3}.$$

Then

$$-\frac{1}{2\pi^2 K(x_3)}\operatorname{Im}\frac{K(w)}{w-x_3} = \frac{\psi(w)-\psi(\overline{w})}{2i},\ w = -i\sqrt{u^2+\alpha^2}+y_3.$$

and

$$\frac{1}{2\pi^2}\Delta_y\left[F(x,y,u)\right] = \frac{1}{2i}\Delta_y\left[\frac{\psi(w)-\psi(\overline{w})}{\sqrt{u^2+\alpha^2}}\right].$$

The function under the sign of the Laplace operator Δ_y depends on $s=\alpha^2$ and y_3, i.e., the function depends on the point $t=(s,y_3)$.

A simple calculation shows that the Laplace operator in the coordinates of the point t has the form

$$\Delta_t = 4s\frac{d^2}{ds^2} + 4\frac{d}{ds} + \frac{d^2}{dy_3^2} = 0. \tag{22}$$

In this way,

$$\frac{1}{2\pi^2}\int_0^\infty \Delta_y\left[F(x,y,u)\right]\cos\lambda u\, du = \frac{1}{2i}\left[\int_0^\infty \cos\lambda u\, \Delta_t\left(\frac{\psi(w)}{\sqrt{u^2+\alpha^2}}\right)du - \int_0^\infty \cos\lambda u\, \Delta_t\left(\frac{\psi(\overline{w})}{\sqrt{u^2+\alpha^2}}\right)du\right].$$

Now, formula (21) follows from the equalities

$$\begin{gathered}\int_0^\infty \cos\lambda u\, \Delta_t\left(\frac{\psi(w)}{\sqrt{u^2+\alpha^2}}\right)du = \lambda^2\int_0^\infty \frac{\psi(w)}{\sqrt{u^2+\alpha^2}}\cos\lambda u\, du, \\ \int_0^\infty \cos\lambda u\, \Delta_t\left(\frac{\psi(\overline{w})}{\sqrt{u^2+\alpha^2}}\right)du = \lambda^2\int_0^\infty \frac{\psi(\overline{w})}{\sqrt{u^2+\alpha^2}}\cos\lambda u\, du.\end{gathered} \tag{23}$$

and formulas (17).

Let us prove the first equality (23). Let I denote the left side of equality (23). Taking into account (22), we obtain

$$I \equiv \int_0^\infty \Delta_t\left(\frac{\psi(w)}{\sqrt{u^2+\alpha^2}}\right)\cos\lambda u\, du = 4s\int_0^\infty \frac{d^2}{ds^2}\left[\frac{\psi(w)}{\sqrt{u^2+\alpha^2}}\right]\cos\lambda u\, du +$$

$$+4\int_0^\infty \frac{d}{ds}\left[\frac{\psi(w)}{\sqrt{u^2+\alpha^2}}\right]\cos\lambda u\, du + \int_0^\infty \frac{d^2}{dy_3^2}\left[\frac{\psi(w)}{\sqrt{u^2+\alpha^2}}\right]\cos\lambda u\, du.$$

Differentiating this equality, we obtain

$$4\frac{d}{ds}\frac{\psi(w)}{\sqrt{u^2+\alpha^2}} = 4\left[\frac{i\psi'(w)}{2(u^2+s)} - \frac{\psi(w)}{2(u^2+s)\sqrt{u^2+\alpha^2}}\right],$$

$$4\frac{d^2}{ds^2}\frac{\psi(w)}{\sqrt{u^2+\alpha^2}} = 4s\left[-\frac{\psi''(w)}{4(u^2+s)^{3/2}} - \frac{i\psi'(w)}{2(u^2+s)^2} - \frac{i\psi'(w)}{4(u^2+s)^2} + \frac{3\psi(w)}{4(u^2+s)\sqrt{u^2+\alpha^2}}\right],$$

$$\frac{d^2}{dy_3^2}\left[\frac{\psi(w)}{\sqrt{u^2+s}}\right] = \frac{\psi''(w)}{\sqrt{u^2+s}}.$$

Grouping the coefficients and adding up the obtained equalities, we obtain

$$I \equiv \int_0^\infty \frac{u^2\psi''(w)\cos\lambda u}{(u^2+s)^{3/2}}du + \int_0^\infty \frac{(2u^2-s)i}{(u^2+s)^2}\psi'(w)\cos\lambda u\,du +$$

$$+\int_0^\infty \frac{s-2u^2}{(u^2+s)^{5/2}}\psi(w)\cos\lambda u\,du = I_1 + I_2 + I_3.$$

As

$$d_u\psi'(w) = \frac{iu\psi''(w)}{\sqrt{u^2+s}}du\,,$$

then integrating the first integral in parts, we obtain

$$I_1 = i\int_0^\infty \left(\frac{s-u^2}{(u^2+s)^2}\cos\lambda u - \frac{u\lambda\sin\lambda u}{u^2+s}\right)\psi'(w)du\,.$$

Hence,

$$I_1 + I_2 = i\int_0^\infty \frac{u^2\psi'(w)}{(u^2+s)^2}\cos\lambda u\,du - i\int_0^\infty \frac{\psi'(w)u\lambda\sin\lambda u}{u^2+s}du\,.$$

We integrate by parts these integrals, where

$$d_u\psi(w) = \frac{iu\psi'(w)}{\sqrt{u^2+s}}du\,.$$

Then the first integral will be equal

$$i\int_0^\infty \frac{u^2\psi'(w)}{(u^2+s)^2}\cos\lambda u\,du =$$

$$= -\int_0^\infty \left[\left(\frac{1}{(u^2+s)^{3/2}} - \frac{3u^2}{(u^2+s)^{5/2}}\right)\cos\lambda u - \frac{u\lambda\sin\lambda u}{(u^2+s)^{3/2}}\right]\psi(w)du,$$

second

$$i\int_0^\infty \frac{\psi'(w)u\lambda\sin\lambda u}{u^2+s} = \int_0^\infty \left[-\frac{u\lambda\sin\lambda u}{(u^2+s)^{3/2}} + \frac{\lambda^2\cos\lambda u}{\sqrt{u^2+s}}\right]\psi(w)du\,.$$

In this way,

$$I_1 + I_2 = \lambda^2\int_0^\infty \psi(w)\frac{\cos\lambda u}{\sqrt{u^2+s}}du - \int_0^\infty \frac{s-u^2}{(u^2+s)^{5/2}}\psi(w)\cos\lambda u\,du\,.$$

Adding the resulting integrals, we obtain

$$I = I_1 + I_2 + I_3 = \lambda^2 \int_0^\infty \psi(w) \frac{\cos \lambda u}{\sqrt{u^2 + s}} du .$$

Thus, the function $\psi(\overline{w})$ is obtained from $\psi(w)$ when replacing i by $-i$; therefore, the calculation does not change and the second equality (23) is proved similarly. The lemma 3 is proved.

Lemma 4. *For $y \neq x$, the following equality holds:*

$$\Delta_y \Phi(x, y) + \lambda^2 \Phi(x, y) = 0 . \tag{24}$$

Proof. We apply the Laplace operator with respect to the variable y in the formula (18). Differentiation is taken under the sign of the integral by virtue of $y \neq x$ and $\alpha > 0$.

As a result, we get

$$2\pi^2 \Delta_y \Phi(x, y) = \int_0^\infty \Delta_y F(x, y, u) \cos \lambda u du,$$

Now equalities (24) follow from (21). The lemma 4 is proved.

Corollary 1. *The function $\Phi(x, y)$ is a fundamental solution of equation (15), and $g(x, y)$ from equality (20) is a regular solution of equation (15) in R^3, including the point $y = x$.*

4. Conclusion

In the present paper, fundamental solutions are constructed for the Helmholtz equation in bounded spaces R^2 and R^3. Fundamental solutions allow us to construct approximate solutions for the Helmholtz equation in the future. The essence of constructing a fundamental solution of the Helmholtz equation is that in the future, using this technique, we will explicitly find a regularized solution of the Helmholtz equation in multidimensional bounded and unbounded domains. The results obtained are used to solve the ill-posed Cauchy problem for the Helmholtz equation.

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Conflicts of interest

The author declares no conflicts of interest.

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