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# On Rational and Irrational Values of Trigonometric Functions

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**Abstract**

The beginning of trigonometry goes back 4000 years from today. Today, there are Euclidean and Non-Euclidean trigonometry. It can be said that trigonometry emerged from the need to make maps containing the position information of the stars, to determine the time and to make a calendar, mostly in astronomy. Each of the six trigonometric functions is defined according to the directed plane angle. For each angle in the domain of these functions, their values correspond to a real number consisting of rational or irrational numbers. Trigonometric functions have an important position in basic sciences and technology as well as calculations in engineering and architecture. In addition to being a branch of mathematics, trigonometry is widely used in solving geometry and analysis problems. It has an indispensable importance in engineering and architectural design and calculations. The values of trigonometric functions are usually an irrational number, with the exception of some special angle values. Irrational numbers are numbers that are not proportional. In other words, they are numbers whose results are uncertain. Examples are numbers such as pi, e, and radical. The irrational values of trigonometric functions, which are infinite decimal expansions, are given in the tables by rounding them to only four or six digits. When entering the trigonometric function values in the tables (or the values obtained in the libraries of electronic calculators) into algebraic operations, the resulting numbers are approximated. In the article, besides showing that most of the trigonometric function values such as sinθ, cosθ and tanθ of many θ angles are irrational, the effect of these functions on the result values of calculations made in engineering and architecture is interpreted.

## 1. Introduction

In ancient Babylon, geometry arose from the practical needs of rulers, cartographers, and builders (Britton et al. 2011). From measurements of fields, walls, posts, buildings, gardens, canals, and ziggurats (temple tower of the city god in Mesopotamia), squares, rectangles, trapezoids, and right triangles were used as the main types of practical geometric shapes (Mansfield and Wildberger, 2017). Ratio-based measurements were an important measurement in Egyptian architecture used to describe the reverse slope (cotθ) of the pyramids.

Evidence that the foundations of trigonometry date back to the ancient Babylonian period between the 19th and 16th centuries BC was reached by deciphering the Plimpton 322 (P322) tablet, the most advanced scientific work of antiquity. The P322 tablet is a trigonometric table thousands of years ahead of its time (Mansfield and Wildberger, 2017).

The table of values of modern trigonometric functions consists of rational and / or irrational ratios of trigonometric functions sinθ, cosθ and tanθ, which show the ratio of the side lengths of right triangles with hypotenuse side length, 1 unit.

The only correct trigonometric table of the world is the trigonometric table of the astronomer and mathematician Hipparchus (190-120 BC) (Mansfield and Wildberger, 2017). The pioneer of studies on trigonometry was Hipparchus, and her successor was Ptolemy (85-160 AD) (Stewart, 2009. The first person whose trigonometric table has survived is Alexandrian

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astronomer Claudius Ptolemaius (Ptolemy), who lived in the 2nd century AD (Van Brummelen, 2012). Ptolemy wrote a 13-volume book on astronomy called Mathēmatikē Syntaxis. This book has been translated into Arabic as the Almagest. In the first volume of the book, the relationship between trigonometry and astronomy was examined, and he arranged the chords ruler with the method he developed for calculating the chords of angles (Gölgeleyen ve Akarsu, 2022). He calculated the approximate value of the beam corresponding to an angle of one degree. The table of values of the sine trigonometric function is included in the book.

It can be said that the need for table of values of trigonometric functions is in astronomical observations and calendar making works rather than solving geometry problems. Each of the six trigonometric functions, consisting of three basic (sinθ, cosθ, tanθ) and three auxiliary (cscθ, secθ, cotθ) trigonometric functions, is defined according to the directed plane angle. For each angle in the domain of these functions, their values in the value set correspond to a real number consisting of rational and/or irrational numbers. The first discovered irrational number is $\sqrt{2}$. This discovery was made because of the curiosity of Greek geometers to measure the diagonal length of a square with a side length of 1 unit with a rational number. With the Pythagorean theorem, it has been proved that there is no rational number satisfying x^2=1+1=2. In the decimal notation of $\sqrt{2}$, it is followed by an infinite number of digits after the decimal symbol. Likewise, today, π has been calculated to tens of trillions of digits after the decimal point. However, the basic digits of the decimal number system used today are based on the Dedekind-Peano axioms system for natural numbers. By expanding the natural numbers axioms system, real and complex number systems are obtained. Although the decimal representation of every rational number consists of a limited number of digits, this expansion continues indefinitely for irrational numbers.

In the article, besides showing that most of the basic trigonometric function values such as sinθ, cosθ and tanθ of many θ plane angles are irrational, the effect of these values on the result values of calculations made in science is examined.

## 2. Method

### 2.1. Some trigonometric identities and equations

For plane angles α and β,

$$\begin{aligned} cos(\alpha+\beta) &= cos\alpha\,.cos\beta - sin\alpha\,.sin\beta \\ sin(\alpha+\beta) &= sin\alpha\,.cos\beta + cos\alpha\,.sin\beta \end{aligned} \tag{1}$$

$$\begin{aligned} cos(\alpha-\beta) &= cos\alpha\,.cos\beta + sin\alpha\,.sin\beta \\ sin(\alpha-\beta) &= sin\alpha\,.cos\beta - cos\alpha\,.sin\beta \end{aligned} \tag{2}$$

Matrices enable us to write them in a compact form,

$$\begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix}\begin{bmatrix} cos\beta & sin\beta \\ -sin\beta & cos\beta \end{bmatrix} = \begin{bmatrix} cos\,(\alpha+\beta) & sin\,(\alpha+\beta) \\ -sin\,(\alpha+\beta) & cos\,(\alpha+\beta \end{bmatrix} \tag{1a}$$

$$\begin{bmatrix} cos\alpha & sin\alpha \\ sin\alpha & -cos\alpha \end{bmatrix}\begin{bmatrix} cos\beta & sin\beta \\ sin\beta & -cos\beta \end{bmatrix} = \begin{bmatrix} \cos\,(\alpha-\beta) & \sin\,(\alpha-\beta) \\ \sin\,(\alpha-\beta) & \cos\,(\alpha-\beta \end{bmatrix} \tag{2a}$$

equations can be written (Beasley, 1858; Dörrie, 1950; Daut, 1951; Helton, 1972; Durell, 1975; Sigl, 1977; Barnet, 1991; Akarsu 2005; Akarsu, 2009).

Also, for any plane angle $\alpha$,

$$sin^2\alpha + cos^2\alpha = 1 \tag{3}$$

the trigonometric equivalent of the Pythagorean theorem can be written.

If α = β = θ is taken and substituted in equations (1) and (2) and equation (3) is taken into account,

$$\begin{aligned} \cos 2\theta = \cos^2\theta - \sin^2\theta &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned} \tag{4}$$

$$\sin 2\theta = 2\sin\theta\,.\cos\theta \tag{5}$$

equations (4) and (5) can be obtained.
Similarly, if $\beta = 2\theta$ is written in equation (1) and equations (3), (4) and (5) are taken into account,

$$\cos 3\theta = 4.\cos^3\theta - 3\cos 3\theta \tag{6}$$

equation (6) can also be written (Beasley,1858; Dörrie, 1950; Daut,1951; Durell, 1975).

$$\begin{aligned} sin2\theta &= 2sin\theta.cos\theta \\ sin3\theta &= 3.sin\theta.cos^2\theta\text{-}sin^3\theta \\ sin4\theta &= 4.sin\theta.cos^3\theta - 4cos\theta.sin^3\theta \\ \sin 5\theta &= 5.\sin\theta - 20.sin^3\theta + 16.sin^5\theta \\ &\vdots \end{aligned} \tag{6a}$$

Or

$$\begin{aligned} sin2\theta &= 2sin\theta.cos\theta \\ sin3\theta &= 2cos\theta.sin2\theta -.sin\theta \\ sin4\theta &= 2cos\theta.sin3\theta - sin2\theta \\ \sin 5\theta &= 2cos\theta.sin4\theta - sin3\theta \\ &\vdots \end{aligned} \tag{6b}$$

$$\begin{aligned} cos2\theta = cos^2\theta - sin^2\theta &= 2.cos^2\theta - 1 \\ &= 1 - 2.sin^2\theta \\ cos3\theta &= cos^3\theta - 3.cos\theta.sin^2\theta \\ cos4\theta &= cos^4\theta - 6.cos^2\theta.sin^2\theta + sin^4\theta \\ \cos 5\theta &= 5.\cos\theta - 20.cos^3\theta + 16.cos^5\theta \\ &\vdots \end{aligned} \tag{6c}$$

or

$$cos2\theta = 2.cos^2\theta - 1$$

$$cos3\theta = 2cos\theta.cos2\theta - cos\theta$$

$$cos4\theta = 2cos\theta.cos3\theta - cos2\theta \quad (6d)$$

$$\cos 5\theta = 2cos\theta.cos4\theta - cos3\theta$$

$$\vdots$$

$$tan2\theta = \frac{2.tan\theta}{1 - tan^2\theta}$$

$$tan3\theta = \frac{3.tan\theta - tan^3\theta}{1 - 3.tan^2\theta} \quad (6e)$$

$$tan4\theta = \frac{4.tan\theta - 4tan^3\theta}{1 - 6.tan^2\theta + tan^4\theta}$$

$$\vdots$$

or

$$tan2\theta = \frac{2.tan\theta}{1 - tan^2\theta}$$

$$tan3\theta = \frac{tan\theta + tan2\theta}{1 - tan\theta.tan2\theta} \quad (6f)$$

$$tan4\theta = \frac{tan\theta + tan3\theta}{1 - tan\theta.tan3\theta}$$

$$\vdots$$

From (6a) to (6f), different trigonometric equations can be written (Sigl, 1977).

## 2.2. Irrationality of Trigonometric Ratios

In this section, the irrationality of the values of the basic trigonometric functions $cos20^{\circ}$, $sin10^{\circ}$ ve $tan15^{\circ}$ of rational angles such as $20^{\circ}, 10^{\circ}$ ve $15^{\circ}$ are shown.

### 2.2.1. Irrationality of Cos20° Trigonometric Function Value

For θ=20° let $cos60^{\circ} = \frac{1}{2}$ and $\cos 20^{\circ} = x$ . If these values are substituted in (6),

$$\cos 60^{\circ} = 4.\cos^3 20^{\circ} - 3.\cos 20^{\circ}$$

$$8x^3 - 6x - 1 = 0 \quad (7)$$

(7) set up the polynomial equation. Because of the way the polynomial equation (7) is set up, $\cos 20^{\circ}$ is a root of the equation. The possible rational roots of equation (7) are $\pm 1, \pm 1/2, \pm 1/4, \pm 1/8$. If these values are substituted in equation (7), it is seen that no root satisfies equation (7).

Since none of the above 8 roots satisfy equation (7), they are not real real roots. Therefore, it is understood that equation (7) has no real roots, so $\cos 20^{\circ}$ is an irrational number (Jung, 1962;Helton, 1972). Whether ±1,±1/(2,±1/4,±1/8 ) rational roots are the real roots of the equation (7) can also be understood by comparing the value of $\cos 20^{\circ} = 0{,}939693$

Namely,

$$\cos 30^{\circ} < \cos 20^{\circ} < \cos 0^{\circ} \text{ or}$$
$$0{,}866025 < \cos 20^{\circ} < 1 \quad (8)$$

(8) inequality can be written.

### 2.2.2. Sin10°Displaying the Irrationality of Trigonometric

**Function Value**

In equation (2), β=2θ is written and equations (3), (4) and (5) are taken into account,

$$\sin 3\theta = 3.\sin\theta - 4.\sin^3\theta \quad (9)$$

equation (9) can be written (Niven, 1956; Jung, 1962; Niven, 1964; Helton, 1972; Bergen, 2009; Paolillo and Vincenzi, 2021).

For $\theta = 10^{\circ}$, let $sin30^{\circ} = \frac{1}{2}$ and s $\sin 10^{\circ} = x$. If these values are substituted in (9),

$$\sin 30^{\circ} = 3.\sin 10^{\circ} - 4.\sin^3 10^{\circ}$$

$$8x^3 - 6x + 1 = 0 \quad (10)$$

(10) polynomial equation can be established.

(4) by virtue of equality,

$$8x^3 - 6x + 1 = 0 \quad (11)$$

(11) equation is taken into account.

Let $\sin 10^{\circ}$ be rational in equation (11). In this case, the expressions $\sin^2 10^{\circ}$and $1 - 2\sin^2 10^{\circ}$ would also be rational. Whereas, c $\cos 20^{\circ}$ was shown to be irrational by equation (7) and inequality (8). Therefore, it is concluded that $\sin 10^{\circ}$is also irrational in order not to cause a contradiction.

### 2.2.3. Demonstrating the Irrationality of the Tan15°Trigonometric Function Value

Starting from equation (4),

$$\tan\theta = \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}} \quad (12)$$

(12) can be written.

$\theta = 15^{\circ}$ için $cos30^{\circ} = \frac{\sqrt{3}}{2}$ dir. For $\theta = 15^{\circ}$, $cos30^{\circ} = \frac{\sqrt{3}}{2}$.

If these values are substituted in the equation (12);

$$tan15^{\circ} = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}} = 0{,}2679491924.$$

Since the value of $tan15^{\circ} = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$ in the square root on the right is irrational, its square root is also irrational. Hence, it is understood that the value of $tan15^{\circ}$ is also irrational. Hence, it is understood that the value of $tan15^{\circ}$ is also irrational.

## 3. Extending the Irrationality of Trigonometric Function Values

It can also be extended outside of some special angles, such as the indicated irrationality of the values of the trigonometric functions $\mathrm{Cos}20^{\circ}$, $\mathrm{Sin}10^{\circ}$ and $\mathrm{Tan}15^{\circ}$. For example, trigonometric function values of angles like $62^{\circ}05'10''$ are irrational. Integer angles such as $0^{\circ}$, $30^{\circ}, 45^{\circ}$ and $60^{\circ}$ and when these angles are added by $90^{\circ}$ and its multiples, or subtracted from $90^{\circ}$ and its multiples, all trigonometric angles The values of the functions cannot be said to be rational. But at least one trigonometric function value is rational.

For example, for all trigonometric function values of $30^{\circ}$,

$$\sin 30^{\circ} = \frac{1}{2}, \csc 30^{\circ} = 2 \text{, rational; } \cos 30^{\circ} = \frac{\sqrt{3}}{2},$$

$$\sec 30^{\circ} = \frac{2}{\sqrt{3}} \text{ irrational; } \tan 30^{\circ} = \frac{\sqrt{3}}{3}, \cot 30^{\circ} = \sqrt{3}$$

irrational, It is seen that 2 trigonometric function values are rational and 4 trigonometric function values are irrational.

Starting from equation (4),

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}, \sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}},$$

$$\tan\theta = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \qquad (13)$$

(13) equations can be written. If angle θ is any angle that makes the trigonometric function value cos2θ irrational, the function values $\sin\theta = f(\cos 2\theta), \cos\theta = f(\cos 2\theta)$ can also be defined using the irrationality of cos2θ to define an infinite number of angles $\theta_i$, $-360^{\circ} \leq \theta_i \leq 360^{\circ}$ the proof that the values of the trigonometric functions in the range of values $-1 \leq \sin\theta, \cos\theta \leq 1$ and $-\infty \leq \tan\theta \leq \infty$, are irrational can be extended. Also, this expansion can be done using the trigonometric equations (6a) to (6f).

$$sin\frac{\theta}{2} = \pm\sqrt{\frac{1-cos\theta}{2}}, cos\frac{\theta}{2} = \pm\sqrt{\frac{1+cos\theta}{2}}$$

The $\theta = 90^{\circ}$ angle can be taken as half of the continuous and the irrational values of the trigonometric functions can be calculated.

$$sin45^{\circ} = \frac{\sqrt{2}}{2}, cos45^{\circ} = \frac{\sqrt{2}}{2}$$

$$sin22{,}5^{\circ} = \frac{1}{2}\sqrt{2-\sqrt{2}}, cos22{,}5^{\circ} = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

$$sin11{,}25^{\circ} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$$

$$cos11{,}25^{\circ} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$sin5{,}625^{\circ} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}+\sqrt{2}}}$$

$$cos5{,}625^{\circ} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}+\sqrt{2}}}$$

irrational values can be calculated.

Similarly, taking $\theta = \frac{90^{\circ}}{2^n}$, $n \in \mathbb{Z}^+$, the values of the trigonometric functions of the angles are,

$$sin\frac{90°}{2^n} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}+\cdots}}}$$

$$cos\frac{90°}{2^n} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}+\cdots}}}$$

it can be calculated by chaining equations (Sigl, 1977).

$cos\theta = t$ olmak üzere,

$$sin\frac{\theta}{2} = \sqrt{\frac{1-t}{2}} = \frac{1}{2}\sqrt{2-2t}, cos\frac{\theta}{2} = \sqrt{\frac{1+t}{2}} = \frac{1}{2}\sqrt{2+2t}$$

$cos\frac{\theta}{2} = \frac{1}{2}\sqrt{2+2t}$ = k including,

$$sin\frac{\theta}{4} = \frac{1}{2}\sqrt{2-2k} = \frac{1}{2}\sqrt{2-\sqrt{2+2t}}$$

$$cos\frac{\theta}{4} = \frac{1}{2}\sqrt{2+2k} = \frac{1}{2}\sqrt{2+\sqrt{2+2t}}$$

equations can be written.

$$cos\frac{\theta}{4}=\frac{1}{2}\sqrt{2+\sqrt{2+2k}}=r$$

$$sin\frac{\theta}{8}=\frac{1}{2}\sqrt{2-2r}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+2t}}}$$

$$cos\frac{\theta}{8}=\frac{1}{2}\sqrt{2+2r}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+2t}}}$$

(14)

$cos\theta=t$ including,

$$sin\frac{\theta}{2}=\frac{1}{2}\sqrt{2-2t}$$

$$sin\frac{\theta}{4}=\frac{1}{2}\sqrt{2-\sqrt{2+2t}}$$

$$sin\frac{\theta}{8}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+2t}}}$$

$$cos\frac{\theta}{2}=\frac{1}{2}\sqrt{2+2t}$$

$$cos\frac{\theta}{4}=\frac{1}{2}\sqrt{2+\sqrt{2+2t}} \qquad (15)$$

$$cos\frac{\theta}{8}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+2t}}}$$

equations (14) and (15) can be written.

Now, using the equations (14), (15) (1) and (2), the irrationality of the sine and cosine trigonometric function values of many angles is shown below.

$$sin30^{\circ}=\tfrac{1}{2}\ ,\ cos30^{\circ}=\tfrac{1}{2}\sqrt{3}$$

$$sin15^{\circ}=\frac{1}{2}\sqrt{2-\sqrt{3}}$$

$$cos15^{\circ}=\frac{1}{2}\sqrt{2+\sqrt{3}}$$

$$sin18^{\circ}=\frac{1}{4}(\sqrt{5}-1)$$

$$cos18^{\circ}=\frac{1}{4}\sqrt{10+2\sqrt{5}}$$

$$sin7{,}5^{\circ}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}$$

$$cos7{,}5^{\circ}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}$$

$$sin3{,}75^{\circ}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$$

$$cos3{,}75^{\circ}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$$

$$sin3^{\circ}=\sin(18^{\circ}-15^{\circ}) = sin18^{\circ}cos15^{\circ}-cos18^{\circ}sin15^{\circ}$$

$$=\frac{1}{16}(\sqrt{30}+\sqrt{10}-\sqrt{6}-\sqrt{2} - 2\sqrt{20+4\sqrt{5}-10\sqrt{3}-2\sqrt{15}})$$

$$sin6^{\circ}=cos84^{\circ}=\sin(36^{\circ}-30^{\circ}) =\frac{1}{8}(\sqrt{30-6\sqrt{5}}-\sqrt{5}-1)$$

$$sin9^{\circ}=cos81^{\circ}=\sin(45^{\circ}-36^{\circ}) =\frac{1}{8}(\sqrt{10}+\sqrt{2}-2\sqrt{5-\sqrt{5}})$$

$$sin12^{\circ}=cos78^{\circ}=\sin(30^{\circ}-18^{\circ}) =\frac{1}{8}(\sqrt{10+2\sqrt{5}}-\sqrt{15}+\sqrt{3})$$

$$sin21^{\circ}=cos69^{\circ}=\sin(36^{\circ}-15^{\circ})$$

$$=\frac{1}{8}(\sqrt{15-3\sqrt{5}}+\sqrt{5-\sqrt{5}}-\sqrt{10-5\sqrt{3}} -\sqrt{2-\sqrt{3}})$$

$$sin24^{\circ}=cos66^{\circ}=\sin(54^{\circ}-30^{\circ})$$

$$=\frac{1}{8}(\sqrt{15}+\sqrt{3}-\sqrt{10-2\sqrt{5}})$$

$$sin27^{\circ}=cos63^{\circ}=\sin(45^{\circ}-18^{\circ})$$

$$=\frac{1}{8}(2\sqrt{5+\sqrt{5}}-\sqrt{10}+\sqrt{2})$$

$$sin33^{\circ}=cos57^{\circ} =\sin(15^{\circ}+18^{\circ}) =\frac{1}{16}(2\sqrt{20+4\sqrt{5}-10\sqrt{3}-2\sqrt{15}}+\sqrt{30}+\sqrt{10} -\sqrt{6}-\sqrt{2})$$

$$sin39^{\circ} = cos51^{\circ} = \sin(54^{\circ} - 15^{\circ}) = \frac{1}{16}(\sqrt{30} + \sqrt{6} + \sqrt{10} + \sqrt{2} - 2.\sqrt{20 + 2\sqrt{15} - 4\sqrt{5} - 10\sqrt{3}})$$

$$sin42^{\circ} = cos48^{\circ} = \sin(60^{\circ} - 18^{\circ}) = \frac{1}{8}(\sqrt{30 + 6\sqrt{5}} - \sqrt{5} + 1)$$

$$sin48^{\circ} = cos42^{\circ} = \sin(30^{\circ} + 18^{\circ}) = \frac{1}{8}(\sqrt{30 + 2\sqrt{5}} + \sqrt{15} - \sqrt{3})$$

$$sin51^{\circ} = cos39^{\circ} = \sin(36^{\circ} + 15^{\circ}) = \frac{1}{8}(\sqrt{15 - 3\sqrt{5}} - \sqrt{5 - \sqrt{5}} + \sqrt{10 - 5\sqrt{3}} + \sqrt{2 - \sqrt{3}})$$

$$sin57^{\circ} = cos33^{\circ} = \sin(72^{\circ} - 15^{\circ}) = \frac{1}{8}(\sqrt{15 + 3\sqrt{5}} - \sqrt{5 + \sqrt{5}} - \sqrt{10 - 5\sqrt{3}} + \sqrt{2 - \sqrt{3}})$$

$$sin63^{\circ} = cos27^{\circ} = \sin(45^{\circ} + 18^{\circ}) = \frac{1}{8}(2.\sqrt{5 + \sqrt{5}} + \sqrt{10} - \sqrt{2})$$

$$sin66^{\circ} = cos24^{\circ} = \sin(36^{\circ} + 30^{\circ}) = \frac{1}{8}(\sqrt{30 - 6\sqrt{5}} + \sqrt{5} + 1)$$

$$sin69^{\circ} = cos21^{\circ} = \sin(54^{\circ} + 15^{\circ}) = \frac{1}{16}(\sqrt{30} + \sqrt{10} + \sqrt{6} + \sqrt{2} + 2.\sqrt{20 + 2\sqrt{15} - 4\sqrt{5} - 10\sqrt{3}})$$

$$sin78^{\circ} = cos12^{\circ} = \sin(60^{\circ} + 18^{\circ}) = \frac{1}{8}(\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1)$$

$$sin81^{\circ} = cos9^{\circ} = \sin(45^{\circ} + 36^{\circ}) = \frac{1}{8}(\sqrt{10} + \sqrt{2} + 2.\sqrt{5 - \sqrt{5}})$$

$$sin84^{\circ} = cos6^{\circ} = \sin(54^{\circ} + 30^{\circ}) = \frac{1}{8}(\sqrt{10 - 2\sqrt{5}} + \sqrt{15} + \sqrt{3})$$

$$sin87^{\circ} = cos3^{\circ} = \sin(72^{\circ} + 15^{\circ}) = \frac{1}{8}(\sqrt{15 + 3\sqrt{5}} + \sqrt{5 + \sqrt{5}} + \sqrt{10 - 5\sqrt{3}} - \sqrt{2 - \sqrt{3}})$$

As shown in the above equations, $3^{\circ}, 6^{\circ}, 9^{\circ}, 12^{\circ}, 15^{\circ}, 18^{\circ}, 21^{\circ}, 24^{\circ}, 27^{\circ}, 33^{\circ}, 39^{\circ}, 42^{\circ}, 48^{\circ}, 51^{\circ}, 57^{\circ}, 63^{\circ}, 66^{\circ}, 69^{\circ}, 78^{\circ}, 81^{\circ}, 84^{\circ}, 87^{\circ}$ sine and cosine trigonometric function values of angles, $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{10}, \sqrt{15}, \sqrt{30}$ as they are calculated based on irrational numbers, their results are irrational (Beasley, 1858; Dörrie, 1950; Daut, 1951; Niven, 1956; Jung, 1962; Niven, 1964; Helton, 1972; Durell, 1975; Sigl, 1977; Barnet, 1991; Akarsu 2005; Akarsu, 2009; Bergen, 2009; Paolillo and Vincenzi, 2021). The rational and irrational values of trigonometric functions are used in applications of engineering sciences (Maune, 2001; Yakar, 2011; Pellicani et al. 2016; Alptekin et al. 2019; Ulvi et al., 2019; Alptekin, 2020).

## 4. Results and Discussion

Trigonometric functions are periodic functions with important applications in mathematics, science and technology. Trigonometric functions are widely used in science. Although rational numbers are numbers with finite decimal expansions, irrational numbers have infinite or noncyclic expansions. It has been shown by equations (13) that the trigonometric functions have infinitely many values in the 0^°≤α≤360^° definition range, while they have infinitely many irrational values in the -1≤sinα,cosα≤1 and-∞≤tanα≤∞ value ranges. As it can be understood from the examples given, it is understood from the examples (2.2.1), (2.2.2) and (2.2.3) above that although the rational values of the trigonometric functions in the value set are quite few, the majority of them consist of irrational numbers. Intermediate calculation values using trigonometric functions in basic sciences, astronomy, engineering and architecture, and technology contain rounding errors. Therefore, it is thought that it is important to be aware that the results of the calculations made with the irrational values of the trigonometric functions used in science always contain an error. It can be said that trigonometric functions, which are a simple geometric way of thinking, contribute to the development of today's science, mathematics, engineering and architectural education.

### Author contributions

The article is single-authored.

### Conflicts of interest

The authors declare no conflicts of interest.

### Statement of Research and Publication Ethics

Research and publication ethics were complied with in the study.

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